Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 344 (2008) 627–641

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Finite element error analysis for a projection-based variational multiscale method with nonlinear eddy viscosity

V. John [a], S. Kaya [b], A. Kindl [a,*]

[a] Universität des Saarlandes, FR 6.1 Mathematik, Postfach 15 11 10, 66041 Saarbrücken, Germany
[b] Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey

Received 8 February 2007

Available online 16 March 2008

Submitted by M.D. Gunzburger

Abstract

The paper presents a finite element error analysis for a projection-based variational multiscale (VMS) method for the incompressible Navier–Stokes equations. In the VMS method, the influence of the unresolved scales onto the resolved small scales is modeled by a Smagorinsky-type turbulent viscosity.

Keywords: Turbulent incompressible flows; Projection-based variational multiscale methods; Finite element error analysis

1. Introduction

Incompressible flows are modeled by the incompressible Navier–Stokes equations which read in dimensionless form

$$\begin{aligned}
&\mathbf{u}_t - 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } (0,T]\times\Omega,\\
&\nabla\cdot\mathbf{u} = 0 \quad \text{in } [0,T]\times\Omega,\\
&\mathbf{u} = \mathbf{0} \quad \text{in } [0,T]\times\partial\Omega,\\
&\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0 \quad \text{in } \Omega,\\
&\int_\Omega p\,d\mathbf{x} = 0 \quad \text{in } (0,T].
\end{aligned} \tag{1.1}$$

Here, $\mathbf{u}$ is the fluid velocity, $\mathbb{D}(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$ is the velocity deformation tensor (symmetric part of the gradient), p is the pressure, $\mathbf{f}$ is an external force, ν is the kinematic viscosity, $\mathbf{u}_0$ is the initial velocity field, $\Omega\subset\mathbb{R}^d$

* Corresponding author.
E-mail addresses: john@math.uni-sb.de (V. John), smerdan@metu.edu.tr (S. Kaya), adela@c-kindl.de (A. Kindl).
URLs: http://www.math.uni-sb.de/ag/john/ (V. John), http://www.metu.edu.tr/~smerdan/ (S. Kaya), http://www.math.uni-sb.de/ag/john/AG/ag.html (A. Kindl).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.03.015

($d = 2$ or $d = 3$) is a bounded, connected domain with polygonal boundary $\partial\Omega$, and $[0, T]$ is a finite time interval. Given a characteristic length scale L and velocity scale U, the Reynolds number is defined by $Re = UL/\nu$.

We are interested in the simulation of turbulent flows which are characterized by a large Reynolds number. For physical reasons (vortex stretching), the case $d = 3$ is of main interest for such flows [36]. From the physical point of view, turbulent flows are characterized by a huge range of scales, reaching from very small ones ($\mathcal{O}(Re^{-9/4})$) to large ones ($\mathcal{O}(|\Omega|)$). A standard discretization of (1.1), like the Galerkin finite element method (FEM), seeks to simulate all persistent scales. However, in particular in three dimensions, even with the present day range of computer memory it is not possible to use meshes which are able to resolve the smallest scales. Consequently, it is not feasible to simulate them.

A natural idea consists in trying to simulate only the behavior of large scales accurately. There are essentially two approaches in this direction, which differ in some important features. The first one is the classical Large Eddy Simulation (LES). In classical LES, the large scales are defined by an average in space (convolution with a filter function) and the influence of the nonresolved small scales onto all large scales is described by a turbulence model. Numerical analysis results for classical LES models are available, see [3,20,22,29]. However, the classical LES approach possesses some drawbacks like commutation errors [2,4,8,40]. In addition, the question of appropriate boundary conditions for the large scales is not solved, see [35].

An alternative approach for the simulation of the large scales are Variational Multiscale (VMS) methods, based on general ideas from [15,17]. The first application of these ideas to turbulent flow problems can be found in [16]. In VMS methods, the large scales are defined by projections into appropriate function spaces. There are classes of VMS methods which rely on a three-scale decomposition of the flow field into large, resolved small and unresolved scales [7]. In contrast to the traditional LES, the influence of the unresolved small scales is described by a model which acts directly only on the resolved small scales (and not on all resolved scales). VMS methods have been proven superior to traditional LES methods in a number of numerical studies, see for instance [11,18,19,25,31,37].

There are different realizations of VMS methods within the framework of FEM, see [27]. We will consider in this paper a so-called projection-based VMS method which possesses the following parameters:

- a velocity and a pressure finite element space (X^h, Q^h) for all resolved scales,
- a large scale velocity finite element space V^H,
- a turbulence model ν_T (turbulence viscosity) describing the direct influence of the unresolved scales onto the resolved small scales.

This method has some similarities but also a number of differences to the projection-based VMS method for which a finite element error analysis was presented in [26]. The two main differences are the definition of the large scales by L^2-projection (instead of elliptic projection in [26]) and the consideration of a nonconstant (even nonlinear) turbulent viscosity ν_T in the present paper.

In the present paper, the turbulent viscosity ν_T depends on the computed solution. This way, one obtains an additional nonlinear term in the equations. Since almost all numerical simulations with VMS methods use Smagorinsky-type models for ν_T, for instance [11,18,19,25,28,31,37], we will study a model of this type also here. It is important to make the turbulent viscosity ν_T concrete since one has to deduce regularity assumptions on the solution of the Navier–Stokes equations such that the arising nonlinear term in the VMS method is well defined for conforming FEM. It turns out that with the Smagorinsky model, different regularity assumptions have to be used than with ν_T being a constant.

In Section 2, the projection-based VMS method is introduced and known analytical results which are used in the error analysis are provided in Section 3. The finite element error analysis is presented in Section 4. Finally, Section 5 contains a numerical illustration which shows the convergence of the projection-based VMS method as the mesh width decreases and the stability of the method as the viscosity becomes small.

2. A projection-based finite element variational multiscale (FEVMS) method

Standard notations are used for Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{m,p}(\Omega)$, $p \in [1, \infty]$, $m \in \mathbb{R}$. The corresponding norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$, respectively. Some norms will be defined in slightly nonstandard ways, see below. The $(L^2(\Omega))^d$ inner product is denoted by $(\cdot,\cdot)$. The same symbol is used for the dual pairing. Let X be a normed space with functions defined in Ω, then $L^p(0,t;X)$ is the space of all functions defined on $(0,t)\times\Omega$ for which the norm

$$\|\mathbf{u}\|_{L^p(0,t;X)} = \left(\int_0^t \|\mathbf{u}\|_X^p\,d\tau\right)^{1/p}, \quad p\in[1,\infty),$$

is finite. For $p=\infty$, the usual modification is used in the definition of this space. The symbol C will be used as a generic positive constant and may have different values at different places, but it is always independent of the viscosity ν and the characteristic size of finite element meshes.

Let

$$X := W_0^{1,2}(\Omega) := \{\mathbf{v}\colon\ \mathbf{v}\in (W^{1,2}(\Omega))^d,\ \mathbf{v}=\mathbf{0} \text{ on } \partial\Omega\},$$
$$Q := L_0^2(\Omega) := \left\{q\in L^2(\Omega)\colon\ \int_\Omega q\,d\mathbf{x}=0\right\}.$$

We consider the following variational formulation of the Navier–Stokes equations (1.1): find $\mathbf{u}:[0,T]\to X$, $p:(0,T]\to Q$ satisfying

$$(\mathbf{u}_t,\mathbf{v}) + (2\nu\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v})) + b_s(\mathbf{u},\mathbf{u},\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) = (\mathbf{f},\mathbf{v}),$$
$$(q,\nabla\cdot\mathbf{u}) = 0, \tag{2.1}$$

for all $(\mathbf{v},q)\in(X,Q)$. Here,

$$b_s(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w}) - \frac{1}{2}((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})$$

is the skew-symmetric trilinear form of the convective term. Obviously, integration by parts shows $b_s(\mathbf{u},\mathbf{v},\mathbf{w}) = -b_s(\mathbf{u},\mathbf{w},\mathbf{v})$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in X$ and consequently

$$b_s(\mathbf{u},\mathbf{v},\mathbf{v}) = 0 \quad \forall\mathbf{u},\mathbf{v}\in X. \tag{2.2}$$

Let

$$V = \{\mathbf{v}\in X\colon\ (\nabla\cdot\mathbf{v},q)=0,\ \forall q\in Q\},$$

then (2.1) can be reformulated in the space of weakly divergence-free functions: find $\mathbf{u}:[0,T]\to V$ satisfying

$$(\mathbf{u}_t,\mathbf{v}) + (2\nu\mathbb{D}(\mathbf{u}),\mathbb{D}(\mathbf{v})) + b_s(\mathbf{u},\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}) \tag{2.3}$$

for all $\mathbf{v}\in V$.

The existence of a solution of (2.3) is known, but not its uniqueness in the case of a three-dimensional domain Ω, e.g. see [10,39]. However, for the finite element error analysis, we have to increase the regularity assumptions on the solution of (2.3), see (4.1). The higher regularity assumptions will imply the uniqueness of the solution.

As explained in the introduction, a standard Galerkin finite element discretization of (2.3) is not feasible in the case of turbulent flows. We will consider a FEVMS method. Let $\mathcal{T}^h$ be an admissible triangulation of Ω in the usual sense, [6], with the mesh cell parameter $h>0$ (maximum of the diameters of the mesh cells) and let a three scale decomposition of the flow field be given by

$$\mathbf{v} = \bar{\mathbf{v}}^h + \tilde{\mathbf{v}}^h + \hat{\mathbf{v}} \quad \text{for all } \mathbf{v}\in X,$$
$$q = \bar{q}^h + \tilde{q}^h + \hat{q} \quad \text{for all } q\in Q,$$

where appropriate finite element spaces for $\bar{\mathbf{v}}^h, \tilde{\mathbf{v}}^h, \bar{q}^h, \tilde{q}^h$ have to be defined. There are several ways for the definition of these spaces [27]. In a so-called bubble FEVMS method, standard finite element spaces (with respect to the discretization of the Navier–Stokes equations) are used for the large scales $(\bar{\mathbf{v}}^h, \bar{q}^h)$. The resolved small scales $(\tilde{\mathbf{v}}^h, \tilde{q}^h)$ are approximated by localized finite element functions, so-called bubble functions. A projection-based FEVMS approach uses standard velocity-pressure finite element spaces (X^h, Q^h) for all resolved scales $(\bar{\mathbf{v}}^h+\tilde{\mathbf{v}}^h, \bar{q}^h+\tilde{q}^h)$ and an additional space V^H to define the large scales. Lastly, one can exploit the hierarchy of basis functions in hierarchical finite element methods for defining large and resolved small scales [21].

This paper considers a continuous-in-time projection-based FEVMS method with conforming and inf–sup stable finite element spaces for the resolved scales, i.e. $X^h \subset X$, $Q^h \subset Q$ and

$$\inf_{q^h\in Q^h}\sup_{\mathbf{v}^h\in X^h}\frac{(q^h,\nabla\cdot\mathbf{v}^h)}{\|q^h\|_{L^2}\|\nabla\mathbf{v}^h\|_{L^2}}\geqslant\beta>0, \tag{2.4}$$

where β is independent of h. Because of the appearance of a nonlinear viscous term of Smagorinsky-type in the FEVMS method, we have to require even $X^h \subset (W_0^{1,3}(\Omega))^d$, see below. Under the inf–sup condition (2.4), the space of discretely divergence-free functions

$$V^h := \{\mathbf{v}^h \in X^h:\ (\nabla\cdot\mathbf{v}^h, q^h) = 0,\ \forall q^h \in Q^h\}$$

is not empty and the Galerkin finite element method of (2.3) can be written as follows: find $\mathbf{u}^h : [0,T] \to V^h$ such that

$$(\mathbf{u}_t^h,\mathbf{v}^h)+(2\nu\mathbb{D}(\mathbf{u}^h),\mathbb{D}(\mathbf{v}^h))+b_s(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h)=(\mathbf{f},\mathbf{v}^h) \tag{2.5}$$

for all $\mathbf{v}^h \in V^h$. A projection-based FEVMS method introduces on the left-hand side of (2.5) an additional viscous term which acts directly only on the resolved small scales: find $\mathbf{u}^h = \bar{\mathbf{u}}^h + \tilde{\mathbf{u}}^h : [0,T] \to V^h$ such that

$$(\mathbf{u}_t^h,\mathbf{v}^h)+(2\nu\mathbb{D}(\mathbf{u}^h),\mathbb{D}(\mathbf{v}^h))+b_s(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h)+(\nu_T(\mathbb{D}(\tilde{\mathbf{u}}^h))\mathbb{D}(\tilde{\mathbf{u}}^h),\mathbb{D}(\tilde{\mathbf{v}}^h))=(\mathbf{f},\mathbf{v}^h) \tag{2.6}$$

for all $\mathbf{v}^h = \bar{\mathbf{v}}^h + \tilde{\mathbf{v}}^h \in V^h$. We consider in this paper a turbulent viscosity of Smagorinsky-type

$$\nu_T(\mathbb{D}(\tilde{\mathbf{u}}^h)) = \begin{cases} C_S\delta^2|\mathbb{D}(\tilde{\mathbf{u}}^h)| & \text{if } C_S\delta^2|\mathbb{D}(\tilde{\mathbf{u}}^h)| \geqslant \nu_0(h) \geqslant 0,\\ \nu_0(h) & \text{else,}\end{cases} \tag{2.7}$$

where $\nu_0(h)$ is a user-defined parameter with $\nu_0(h) \to 0$ for $h \to 0$, $C_S > 0$, $\delta > 0$ is a quantity which is connected to the resolution of the finite element spaces involved in the VMS method (mesh size h of the fine scales or H of the large scales, see below) and $|\cdot|$ being the Frobenius norm of a tensor. The large scale part $\bar{\mathbf{u}}^h \in V^H$ of the velocity $\mathbf{u}^h$ is defined by an L^2-projection into the large scale space V^H

$$0 = (\mathbf{u}^h - \bar{\mathbf{u}}^h, \mathbf{v}^H) =: ((I - P_H)\mathbf{u}^h, \mathbf{v}^H)$$

for all $\mathbf{v}^H \in V^H \subset L^2(\Omega)$.

Consequently, the resolved small scale part of the velocity is given by

$$\tilde{\mathbf{u}}^h = \mathbf{u}^h - \bar{\mathbf{u}}^h = (I - P_H)\mathbf{u}^h.$$

Remark 2.1. In practice, the Smagorinsky model is used with $\nu_0(h) = 0$. With this choice, it is possible that there are points $\mathbf{x} \in \Omega$ where this model vanishes, namely if $\mathbb{D}(\tilde{\mathbf{u}}^h(\mathbf{x})) = \mathbb{O}$. That means, there are no small scales in a region of the flow. Clearly, in this region, the flow is not turbulent and the use of a turbulence model is not necessary. However, from the analytical point of view, if there is no uniform positive bound form below for the additional viscosity, one cannot obtain different estimates than for the Navier–Stokes equations since the proof uses only global estimates. In the finite element error estimate, the assumption $\nu_0(h) > 0$ will be used for estimating the trilinear term of the Navier–Stokes equations, in particular see estimates (4.18) and (4.19).

3. Preliminaries

This section collects results which will be used in the finite element error analysis.

The Smagorinsky LES model, which possesses the additional nonlinear viscous term $(\nu_T(\mathbb{D}(\mathbf{u}))\mathbb{D}(\mathbf{u}), \mathbb{D}(\mathbf{v}))$, where $\nu_T(\mathbb{D}(\mathbf{u}))$ is given in (2.7) with $\nu_0(h) = 0$, on the left-hand side of the variational formulation (2.3) is well understood. The natural space to study this model is $(W_0^{1,3}(\Omega))^d$. Ladyzhenskaya [32] proved the existence and uniqueness of a weak solution of the Smagorinsky model. Finite element error estimates can be found in [29], see also [22]. It turns out that estimates independent of the Reynolds number can be derived either under the assumption $\nu_0(h) > 0$ or under the assumptions of a higher regularity of the solution and that the corresponding norm of the solution is independent of the Reynolds number. The second approach would also be possible for the VMS method (2.6), but the necessary assumptions are not likely to be fulfilled for real turbulent flow fields.

The most important analytical tools in the analysis of the Smagorinsky model are a strong monotonicity of the nonlinear viscous term: there is a constant $C > 0$ such that for all $\mathbf{u}, \mathbf{v} \in (W^{1,3}(\Omega))^d$, $d \in \{2, 3\}$,

$$\big(|\mathbb{D}(\mathbf{u})|\mathbb{D}(\mathbf{u}) - |\mathbb{D}(\mathbf{v})|\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{u}-\mathbf{v})\big) \geqslant C\|\mathbb{D}(\mathbf{u}-\mathbf{v})\|_{L^3}^3, \tag{3.1}$$

and the local Lipschitz continuity: there is a $C > 0$ such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in (W^{1,3}(\Omega))^d$, $d \in \{2, 3\}$,

$$\big(|\mathbb{D}(\mathbf{u})|\mathbb{D}(\mathbf{u}) - |\mathbb{D}(\mathbf{v})|\mathbb{D}(\mathbf{v}), \mathbb{D}(\mathbf{w})\big) \leqslant CC_L\|\mathbb{D}(\mathbf{u}-\mathbf{v})\|_{L^3}\|\mathbb{D}(\mathbf{w})\|_{L^3}, \tag{3.2}$$

where $C_L = \max\{\|\mathbb{D}(\mathbf{u})\|_{L^3}, \|\mathbb{D}(\mathbf{v})\|_{L^3}\}$. These estimates are also a key in the subsequent analysis.

In the analysis, a precise characterization of the large scale space V^H is necessary. The finite element spaces (V^h, Q^h) contain all resolved scales. The large scale space $V^H \subset (W^{1,3}(\Omega))^d$ may be a finite element space defined on the same grid as V^h, but with functions having a lower piecewise polynomial degree, or on a coarser grid. However, $V^H \not\subset V^h$ since no boundary conditions are included into the definition of V^H. Let Q^H be a coarse pressure finite element space such that (V^H, Q^H) fulfills an inf–sup condition of form (2.4). The large scales $\bar{\mathbf{u}}^h = P_H\mathbf{u}$ of the velocity and the large scales $\bar{p}^h = P_H p$ of the pressure are defined by the L^2-projection into V^H, and Q^H respectively; $P_H : (V, Q) \to (V^H, Q^H)$

$$\big(\mathbf{u} - P_H\mathbf{u}, \mathbf{v}^H\big) = 0 \quad \forall \mathbf{v}^H \in V^H,$$
$$\big(p - P_H p, q^H\big) = 0 \quad \forall q^H \in Q^H.$$

We assume that the finite element spaces V^h, V^H rely on quasiuniform triangulations of Ω such that standard inverse estimates for the finite element functions hold. The inverse estimate for V^H gives

$$\|\mathbb{D}(P_H\boldsymbol{\phi}^h)\|_{L^2} \leqslant CH^{-1}\|P_H\boldsymbol{\phi}^h\|_{L^2} = CH^{-1}\|\bar{\boldsymbol{\phi}}^h\|_{L^2}, \tag{3.3}$$

where H is the mesh parameter connected with V^H.

The finite element error analysis will need the L^2-stability of the projection for functions from V^h, which follows directly from the definition of the L^2-projection:

$$\|P_H\boldsymbol{\phi}^h\|_{L^2} \leqslant C\|\boldsymbol{\phi}^h\|_{L^2} \quad \forall \boldsymbol{\phi}^h \in V^h. \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$\|\mathbb{D}(P_H\boldsymbol{\phi}^h)\|_{L^2} \leqslant CH^{-1}\|\boldsymbol{\phi}^h\|_{L^2} \quad \forall \boldsymbol{\phi}^h \in V^h. \tag{3.5}$$

Remark 3.1. In [25–27], a different approach for the separation of scales is used. The additional large scale space for the velocity is defined as a tensor-valued space $L^H = \mathbb{D}(V^H)$, $L^H \subset \{\mathbb{L} \in (L^2(\Omega))^{d\times d}, \mathbb{L} = \mathbb{L}^T\}$, with (V^H, Q^H) fulfilling an inf–sup condition of the form (2.4). In this case, the large scales of the velocity are defined by an elliptic projection into V^H:

$$\big(\mathbb{D}(\mathbf{u} - P_H\mathbf{u}), \mathbb{D}(\mathbf{v}^H)\big) = 0 \quad \forall \mathbf{v}^H \in V^H,$$
$$(\mathbf{u} - P_H\mathbf{u}, 1) = 0.$$

The difficulty in the analysis of this approach consists in obtaining an estimate similar to (3.5). In [26], the L^2-stability for the elliptic projection was assumed, but a rigorous mathematical proof or a counterexample seems to be an open question. Note that the natural stability $\|\mathbb{D}(P_H\mathbf{u})\| \leqslant \|\mathbb{D}(\mathbf{u})\|$ of the elliptic projection was not exploited in [26].

Next, an inequality is introduced which relates $(L^p(\Omega))^d$-norms of the gradients of finite element functions to the $(L^2(\Omega))^d$-norm of the gradients. There exists a constant $C = C(p)$ such that for $2 \leqslant p < \infty$, $d \in \{2, 3\}$,

$$\|\nabla\mathbf{v}^h\|_{L^p} \leqslant Ch^{\frac{d}{2}(\frac{2-p}{p})}\|\nabla\mathbf{v}^h\|_{L^2} \tag{3.6}$$

for all $\mathbf{v}^h \in X^h$ [33].

Using similar arguments as in [33], it can be also shown that there exists a constant $C = C(p)$ such that for $2 \leqslant p < \infty$, $d \in \{2, 3\}$,

$$\|\mathbf{v}^h\|_{L^p} \leqslant Ch^{\frac{d}{2}(\frac{2-p}{p})}\|\mathbf{v}^h\|_{L^2} \tag{3.7}$$

for all $\mathbf{v}^h \in X^h$.

In the error analysis, the term $\|\mathbb{D}(\mathbf{v}^h)\|_{L^3}$ has to be estimated from above. The triangle inequality gives

$$\|\mathbb{D}(\mathbf{v}^h)\|_{L^3} \leqslant \|\mathbb{D}(\tilde{\mathbf{v}}^h)\|_{L^3} + \|\mathbb{D}(\bar{\mathbf{v}}^h)\|_{L^3}, \quad \bar{\mathbf{v}}^h = P_H\mathbf{v}^h. \tag{3.8}$$

Applying (3.6), the inverse estimate $\|\nabla\bar{\mathbf{v}}^h\|_{L^2} \leqslant CH^{-1}\|\bar{\mathbf{v}}^h\|_{L^2}$ in V^H and the L^2-stability (3.4) leads to

$$\|\mathbb{D}(\bar{\mathbf{v}}^h)\|_{L^3} \leqslant \|\nabla\bar{\mathbf{v}}^h\|_{L^3} \leqslant CH^{-d/6}\|\nabla\bar{\mathbf{v}}^h\|_{L^2} \leqslant CH^{-(d+6)/6}\|\bar{\mathbf{v}}^h\|_{L^2} \leqslant CH^{-(d+6)/6}\|\mathbf{v}^h\|_{L^2}. \tag{3.9}$$

Combining (3.8) and (3.9) gives

$$\|\mathbb{D}(\mathbf{v}^h)\|_{L^3} \leqslant \|\mathbb{D}(\tilde{\mathbf{v}}^h)\|_{L^3} + CH^{-(d+6)/6}\|\mathbf{v}^h\|_{L^2}. \tag{3.10}$$

Finally, standard inequalities which will be used are summarized for the convenience of the reader:

– Young's inequality: for $a, b > 0$,

$$ab \leqslant \frac{\varepsilon}{p}a^p + \frac{\varepsilon^{-q/p}}{q}b^q, \quad 1 < p, q < \infty, \ \frac{1}{p} + \frac{1}{q} = 1, \ \varepsilon > 0,$$

– Poincaré's inequality: for all $\mathbf{v} \in (W_0^{1,2}(\Omega))^d$ holds

$$\|\mathbf{v}\|_{L^2} \leqslant C\|\nabla\mathbf{v}\|_{L^2},$$

– Korn's inequality: for all $\mathbf{v} \in (W_0^{1,p}(\Omega))^d$, $p \in (1, \infty)$, holds

$$\|\nabla\mathbf{v}\|_{L^p} \leqslant C\|\mathbb{D}(\mathbf{v})\|_{L^p}.$$

Thanks to Korn's inequality, we can define a norm in $(W^{-1,3/2}(\Omega))^d$ which is equivalent to the usual one

$$\|\mathbf{f}\|_{W^{-1,3/2}} := \sup_{\mathbf{v}\in(W_0^{1,3})^d} \frac{|(\mathbf{f}, \mathbf{v})|}{\|\mathbb{D}(\mathbf{v})\|_{L^3}}. \tag{3.11}$$

Note that neither Poincaré's, nor Korn's inequality can be applied to the large scale velocities $\bar{\mathbf{v}}^h$ and the resolved small scales velocities $\tilde{\mathbf{v}}^h$ since these functions do not, in general, possess homogeneous Dirichlet boundary conditions.

4. Finite element error analysis

In this section, we state and prove the main result of this paper. At the beginning, a priori error estimates (stability results) are given for the solutions of (2.3) and (2.6), respectively.

Lemma 4.1. *Let* $\mathbf{u}$ *with* $\nabla\mathbf{u} \in L^2(0, T; L^2)$ *and* $\mathbf{u}_t \in L^2(0, T; H^{-1})$ *be a solution of* (2.3) *and let the regularity assumptions* $\mathbf{f} \in L^2(0, T; H^{-1})$, $\mathbf{u}_0 \in (L^2(\Omega))^d$ *on data be fulfilled. Then, it satisfies the stability estimate*

$$\frac{1}{2}\|\mathbf{u}(t)\|_{L^2}^2 + \nu\|\mathbb{D}(\mathbf{u})\|_{L^2(0,t;L^2)}^2 \leqslant \frac{1}{2}\|\mathbf{u}(0)\|_{L^2}^2 + \frac{C}{\nu}\|\mathbf{f}\|_{L^2(0,t;H^{-1})}^2$$

for all $t \in [0, T]$.

Proof. The proof follows from setting $\mathbf{v} = \mathbf{u}$ in (2.3), using (2.2), estimating the dual pairing in the usual way and applying Poincaré's, Korn's and Young's inequality. □

Now, a stability estimate for the solution of the FEVMS method (2.6) is given whose constants on the right-hand side do not depend on ν^{-1}.

Lemma 4.2. *Let $\mathbf{u}^h \in V^h$ be a solution of* (2.6) *with $\mathbf{u}_t \in L^2(0,T;W^{-1,3/2})$ and $\nu_0(h) \geqslant 0$ and let the regularity assumption $\mathbf{f} \in L^2(0,T;W^{-1,3/2})$ be fulfilled. Then the stability bound*

$$\|\mathbf{u}^h(t)\|_{L^2}^2 + 4\nu\|\mathbb{D}(\mathbf{u}^h)\|_{L^2(0,t;L^2)}^2 + \nu_0(h)\|\mathbb{D}(\tilde{\mathbf{u}}^h)\|_{L^2(0,t;L^2)}^2 + \frac{C_T}{2}\delta^2\|\mathbb{D}(\tilde{\mathbf{u}}^h)\|_{L^3(0,t;L^3)}^3$$
$$\leqslant C\exp(t)\big[\|\mathbf{u}^h(0)\|_{L^2}^2 + H^{-(d+6)/3}\|\mathbf{f}\|_{L^2(0,T;W^{-1,3/2})}^2 + \delta^{-1}\|\mathbf{f}\|_{L^{3/2}(0,T;W^{-1,3/2})}^{3/2}\big]$$

holds for $C_T, C > 0$ and all $t \in [0,T]$.

Proof. Setting $\mathbf{v}^h = \mathbf{u}^h$ in (2.6), using the skew-symmetry of the trilinear term and the definition (3.11) of the $(W^{-1,3/2}(\Omega))^d$-norm gives

$$\frac{d}{dt}\|\mathbf{u}^h\|_{L^2}^2 + 4\nu\|\mathbb{D}(\mathbf{u}^h)\|_{L^2}^2 + 2\big(\nu_T(\mathbb{D}(\tilde{\mathbf{u}}^h))\mathbb{D}(\tilde{\mathbf{u}}^h), \mathbb{D}(\tilde{\mathbf{u}}^h)\big) \leqslant 2\|\mathbf{f}\|_{W^{-1,3/2}}\|\mathbb{D}(\mathbf{u}^h)\|_{L^3}.$$

The estimate of the nonlinear viscous term uses (3.1) with $\mathbf{v} = \mathbf{0}$, the nonnegativity of the integrand and that the arithmetic mean of two numbers is less or equal than the larger one

$$\begin{aligned}
2\big(\nu_T(\mathbb{D}(\tilde{\mathbf{u}}^h))\mathbb{D}(\tilde{\mathbf{u}}^h), \mathbb{D}(\tilde{\mathbf{u}}^h)\big) &= 2\big(\max\{\nu_0(h), C_S\delta^2|\mathbb{D}(\tilde{\mathbf{u}}^h)|\}\mathbb{D}(\tilde{\mathbf{u}}^h), \mathbb{D}(\tilde{\mathbf{u}}^h)\big)\\
&= 2\int_\Omega \max\{\nu_0(h), C_S\delta^2|\mathbb{D}(\tilde{\mathbf{u}}^h)|\}|\mathbb{D}(\tilde{\mathbf{u}}^h)|^2\,d\mathbf{x}\\
&\geqslant \big((\nu_0(h) + C_S\delta^2|\mathbb{D}(\tilde{\mathbf{u}}^h)|)\mathbb{D}(\tilde{\mathbf{u}}^h), \mathbb{D}(\tilde{\mathbf{u}}^h)\big)\\
&\geqslant \nu_0(h)\|\mathbb{D}(\tilde{\mathbf{u}}^h)\|_{L^2}^2 + C_T\delta^2\|\mathbb{D}(\tilde{\mathbf{u}}^h)\|_{L^3}^3
\end{aligned}$$

with $C_T = C$, $C_S > 0$. Inserting this estimate, applying estimate (3.10), Young's inequality and hiding the term $\frac{C_T}{2}\delta^2\|\mathbb{D}(\tilde{\mathbf{u}}^h)\|_{L^3}^3$ on the left-hand side gives

$$\frac{d}{dt}\|\mathbf{u}^h\|_{L^2}^2 + 4\nu\|\mathbb{D}(\mathbf{u}^h)\|_{L^2}^2 + \nu_0(h)\|\mathbb{D}(\tilde{\mathbf{u}}^h)\|_{L^2}^2 + \frac{C_T}{2}\delta^2\|\mathbb{D}(\tilde{\mathbf{u}}^h)\|_{L^3}^3$$
$$\leqslant C\big[H^{-(d+6)/3}\|\mathbf{f}\|_{W^{-1,3/2}}^2 + \delta^{-1}\|\mathbf{f}\|_{W^{-1,3/2}}^{3/2}\big] + \|\mathbf{u}^h\|_{L^2}^2.$$

The statement of the lemma is now obtain by applying Gronwall's inequality. □

Remark 4.1. The corresponding estimate for the Smagorinsky LES model [29, (3.5)] does not possess the exponential factor. In addition, since the term $\|\mathbf{f}\|_{W^{-1,3/2}}^2$ does not appear in the stability estimate for the Smagorinsky model, the regularity $\mathbf{f} \in L^{3/2}(0,T;W^{-1,3/2})$ is sufficient. Hence, the stability bound for the FEVMS method with Smagorinsky-type turbulent viscosity is in fact weaker than for the Smagorinsky LES model. This is natural since the turbulent viscosity acts directly on all resolved scales in the Smagorinsky LES model and only on the resolved small scales in the FEVMS method.

Now, the regularity assumptions for the finite element error analysis are stated:

$$\nabla\mathbf{u} \in \big(L^3(0,T;L^3)\big)^{d\times d}, \qquad \mathbf{u}_t, \mathbf{f} \in \big(L^2(0,T;W^{-1,3/2})\big)^d, \qquad p \in L^2(0,T;L^{3/2}). \tag{4.1}$$

From the stability estimate, Lemma 4.2, it follows that

$$\mathbf{u}^h \in \big(L^\infty(0,T;L^2)\big)^d, \qquad \mathbb{D}(\mathbf{u}^h) \in \big(L^2(0,T;L^2)\big)^{d\times d}, \qquad \mathbb{D}(\tilde{\mathbf{u}}^h) \in \big(L^3(0,T;L^3)\big)^{d\times d}. \tag{4.2}$$

For the error analysis we require

$$\mathbb{D}(\mathbf{u}^h) \in \big(L^3(0,T;L^3)\big)^{d\times d}, \tag{4.3}$$

from which it follows that

$$\mathbb{D}(\bar{\mathbf{u}}^h) = \mathbb{D}(\mathbf{u}^h) - \mathbb{D}(\tilde{\mathbf{u}}^h) \in \big(L^3(0,T;L^3)\big)^{d\times d}. \tag{4.4}$$

For the large scale part of the solution of (2.3) we require

$$\mathbb{D}(\bar{\mathbf{u}}) \in \left(L^3\left(0,T;L^3\right)\right)^{d\times d}, \tag{4.5}$$

and consequently

$$\mathbb{D}(\tilde{\mathbf{u}}) \in \left(L^3\left(0,T;L^3\right)\right)^{d\times d}. \tag{4.6}$$

Concerning $\nabla\mathbf{u}$, (4.1) is the natural regularity assumption for the Smagorinsky LES model. From this assumption it follows in particular that even the weak solution of the Navier–Stokes equations in three dimensions is unique since from the Sobolev imbedding theorem $W^{1,3}(\Omega) \hookrightarrow L^9(\Omega)$ it follows that Serrin's condition [38] is fulfilled. The assumptions on $\mathbf{u}_t$ and $\mathbf{f}$ are somewhat stronger than for the Smagorinsky LES model. The reasons are estimates of the form (3.10) which estimate the $L^3(\Omega)$ norm of derivatives of the large scales by the $L^2(\Omega)$ norm of the function and the subsequent application of Young's inequality with $p=q=2$.

The next step in the analysis consists in deriving an error equation. Let $\mathbf{e}=\mathbf{u}-\mathbf{u}^h$ and subtract the FEVMS equation (2.6) from the continuous equation (2.1) for all test functions $\mathbf{v}^h \in V^h \subset X$. This gives, using in addition that the functions from V^h are discretely divergence-free

$$\begin{aligned}&\left(\mathbf{e}_t,\mathbf{v}^h\right)+\left(2\nu\mathbb{D}(\mathbf{e}),\mathbb{D}\left(\mathbf{v}^h\right)\right)+b_s\left(\mathbf{u},\mathbf{u},\mathbf{v}^h\right)-b_s\left(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h\right)-\left(\nu_T\left(\mathbb{D}\left(\tilde{\mathbf{u}}^h\right)\right)\mathbb{D}\left(\tilde{\mathbf{u}}^h\right),\mathbb{D}\left(\tilde{\mathbf{v}}^h\right)\right)\\&\quad-\left(p-q^h,\nabla\cdot\mathbf{v}^h\right)=0\end{aligned} \tag{4.7}$$

for all $(\mathbf{v}^h,q^h) \in V^h \times Q^h$.

Now, the error is decomposed into an interpolation part and a discrete part by $\mathbf{e}=\boldsymbol{\eta}-\boldsymbol{\phi}^h$ where $\boldsymbol{\eta}=\mathbf{u}-\mathbf{u}_I^h$ and $\boldsymbol{\phi}^h=\mathbf{u}^h-\mathbf{u}_I^h$. Herein, $\mathbf{u}_I^h \in V^h$ is an approximation of $\mathbf{u}$ fulfilling certain interpolation estimates. From the linearity of the projection and the differentiation follows $\bar{\boldsymbol{\phi}}^h=\bar{\mathbf{u}}^h-\bar{\mathbf{u}}_I^h$, $\mathbb{D}(\bar{\boldsymbol{\phi}}^h)=\mathbb{D}(\bar{\mathbf{u}}^h)-\mathbb{D}(\bar{\mathbf{u}}_I^h)$ and the same properties for the resolved small scales. Since $V^h \subset (W^{1,3}(\Omega))^d$, we have $\boldsymbol{\eta},\ \boldsymbol{\phi}^h \in (W^{1,3}(\Omega))^d$. With respect to time regularity, we require the same properties for $\boldsymbol{\eta}$ and $\boldsymbol{\phi}^h$ as for $\mathbf{u}$. In particular, the error analysis will need

$$\mathbb{D}(\boldsymbol{\eta}),\mathbb{D}(\tilde{\boldsymbol{\eta}}) \in \left(L^3\left(0,T;L^3\right)\right)^{d\times d}, \qquad \boldsymbol{\eta}_t \in \left(L^2\left(0,T;W^{-1,3/2}\right)\right)^d. \tag{4.8}$$

The error equation (4.7) can be reformulated by using this decomposition and choosing $\mathbf{v}^h=\boldsymbol{\phi}^h \in V^h$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2}^2+2\nu\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2}^2+\left(\nu_T\left(\mathbb{D}\left(\tilde{\mathbf{u}}^h\right)\right)\mathbb{D}\left(\tilde{\mathbf{u}}^h\right),\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right)-\left(\nu_T\left(\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right)\right)\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right),\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right)\\&\quad=\left(\boldsymbol{\eta}_t,\boldsymbol{\phi}^h\right)+\left(2\nu\mathbb{D}(\boldsymbol{\eta}),\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right)+b_s\left(\mathbf{u},\mathbf{u},\boldsymbol{\phi}^h\right)-b_s\left(\mathbf{u}^h,\mathbf{u}^h,\boldsymbol{\phi}^h\right)-\left(p-q^h,\nabla\cdot\boldsymbol{\phi}^h\right)\\&\qquad-\left(\nu_T\left(\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right)\right)\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right),\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right)\end{aligned} \tag{4.9}$$

for all $q^h \in Q^h$.

From the definition (2.7) of the turbulent viscosity follow the estimates

$$\begin{aligned}&\left(\nu_T\left(\mathbb{D}\left(\tilde{\mathbf{u}}^h\right)\right)\mathbb{D}\left(\tilde{\mathbf{u}}^h\right),\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right)-\left(\nu_T\left(\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right)\right)\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right),\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right)\geqslant 2C_1\delta^2\left\|\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right\|_{L^3}^3,\\&\left(\nu_T\left(\mathbb{D}\left(\tilde{\mathbf{u}}^h\right)\right)\mathbb{D}\left(\tilde{\mathbf{u}}^h\right),\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right)-\left(\nu_T\left(\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right)\right)\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right),\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right)\geqslant \nu_0(h)\left\|\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right\|_{L^2}^2,\end{aligned}$$

with $C_1>0$ and where, for the first estimate, (3.1) has been used. Using again that the arithmetic mean of two numbers is less or equal than the larger number, one gets

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{\phi}^h\right\|_{L^2}^2+2\nu\left\|\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right\|_{L^2}^2+\frac{\nu_0(h)}{2}\left\|\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right\|_{L^2}^2+C_1\delta^2\left\|\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right\|_{L^3}^3\\&\quad\leqslant\left|\left(\boldsymbol{\eta}_t,\boldsymbol{\phi}^h\right)\right|+\left|\left(2\nu\mathbb{D}(\boldsymbol{\eta}),\mathbb{D}\left(\boldsymbol{\phi}^h\right)\right)\right|+\left|b_s\left(\mathbf{u},\mathbf{u},\boldsymbol{\phi}^h\right)-b_s\left(\mathbf{u}^h,\mathbf{u}^h,\boldsymbol{\phi}^h\right)\right|+\left|\left(p-q^h,\nabla\cdot\boldsymbol{\phi}^h\right)\right|\\&\qquad+\left|\left(\nu_T\left(\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right)\right)\mathbb{D}\left(\tilde{\mathbf{u}}_I^h\right),\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right)\right|\end{aligned} \tag{4.10}$$

for all $q^h \in Q^h$.

The first term on the right-hand side of (4.10) is estimated with the same technique as the right-hand side in the stability estimate, see proof of Lemma 4.2. One obtains

$$\left|\left(\boldsymbol{\eta}_t,\boldsymbol{\phi}^h\right)\right|\leqslant C\left(H^{-(d+6)/3}\|\boldsymbol{\eta}_t\|_{W^{-1,3/2}}^2+\delta^{-1}\|\boldsymbol{\eta}_t\|_{W^{-1,3/2}}^{3/2}\right)+\frac{C_1\delta^2}{6}\left\|\mathbb{D}\left(\tilde{\boldsymbol{\phi}}^h\right)\right\|_{L^3}^3+\frac{1}{4}\left\|\boldsymbol{\phi}^h\right\|_{L^2}^2. \tag{4.11}$$

The estimate of the second term in (4.10) uses just the Cauchy–Schwarz inequality and Young's inequality

$$\left|\left(2\nu\mathbb{D}(\boldsymbol{\eta}),\mathbb{D}(\boldsymbol{\phi}^h)\right)\right| \leqslant \nu\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^2}^2+\nu\left\|\mathbb{D}(\boldsymbol{\phi}^h)\right\|_{L^2}^2. \tag{4.12}$$

The pressure term in (4.10) will be estimated using Hölder's inequality, the estimate of the norm of the divergence by the same norm of the deformation tensor and (3.10) to yield

$$\begin{aligned}\left|\left(p-q^h,\nabla\cdot\boldsymbol{\phi}^h\right)\right| &\leqslant \left\|p-q^h\right\|_{L^{3/2}}\left\|\nabla\cdot\boldsymbol{\phi}^h\right\|_{L^3}\\ &\leqslant C\left\|p-q^h\right\|_{L^{3/2}}\left\|\mathbb{D}(\boldsymbol{\phi}^h)\right\|_{L^3}\\ &\leqslant C\left(H^{-(d+6)/3}\left\|p-q^h\right\|_{L^{3/2}}^2+\delta^{-1}\left\|p-q^h\right\|_{L^{3/2}}^{3/2}\right)+\frac{C_1\delta^2}{6}\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^3}^3+\frac{1}{4}\left\|\boldsymbol{\phi}^h\right\|_{L^2}^2. \end{aligned}\tag{4.13}$$

The nonlinear viscous term on the right-hand side of (4.10) is estimated by using the triangle inequality, the local Lipschitz continuity (3.2), the Cauchy–Schwarz inequality and Young's inequality

$$\begin{aligned}\left|\left(\nu_T\left(\mathbb{D}(\tilde{\mathbf{u}}_I^h)\right)\mathbb{D}(\tilde{\mathbf{u}}_I^h),\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right)\right| &\leqslant \left|\left(\left(C_S\delta^2\left|\mathbb{D}(\tilde{\mathbf{u}}_I^h)\right|+\nu_0(h)\right)\mathbb{D}(\tilde{\mathbf{u}}_I^h),\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right)\right|\\ &\leqslant \left|C_S\delta^2\left(\left|\mathbb{D}(\tilde{\mathbf{u}}_I^h)\right|\mathbb{D}(\tilde{\mathbf{u}}_I^h)-\left|\mathbb{D}(\tilde{\mathbf{u}})\right|\mathbb{D}(\tilde{\mathbf{u}}),\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right)\right|+\left|C_S\delta^2\left(\left|\mathbb{D}(\tilde{\mathbf{u}})\right|\mathbb{D}(\tilde{\mathbf{u}}),\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right)\right|\\ &\quad+\left|\left(\nu_0(h)\left(\mathbb{D}(\tilde{\mathbf{u}}_I^h)-\mathbb{D}(\tilde{\mathbf{u}})\right),\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right)\right|+\left|\left(\nu_0(h)\mathbb{D}(\tilde{\mathbf{u}}),\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right)\right|\\ &\leqslant \nu_0(h)\left\|\mathbb{D}(\tilde{\boldsymbol{\eta}})\right\|_{L^2}\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^2}+C\tilde{C}_L\delta^2\left\|\mathbb{D}(\tilde{\boldsymbol{\eta}})\right\|_{L^3}\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^3}\\ &\quad+\nu_0(h)\left\|\mathbb{D}(\tilde{\mathbf{u}})\right\|_{L^2}\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^2}+C\delta^2\left\|\mathbb{D}(\tilde{\mathbf{u}})\right\|_{L^3}^2\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^3}\\ &\leqslant 4\nu_0(h)\left\|\mathbb{D}(\tilde{\boldsymbol{\eta}})\right\|_{L^2}^2+C\tilde{C}_L^{3/2}\delta^2\left\|\mathbb{D}(\tilde{\boldsymbol{\eta}})\right\|_{L^3}^{3/2}+4\nu_0(h)\left\|\mathbb{D}(\tilde{\mathbf{u}})\right\|_{L^2}^2+C\delta^2\left\|\mathbb{D}(\tilde{\mathbf{u}})\right\|_{L^3}^3\\ &\quad+\frac{\nu_0(h)}{8}\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^2}^2+\frac{C_1\delta^2}{6}\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^3}^3 \end{aligned}\tag{4.14}$$

with $\tilde{C}_L=\max\{\|\mathbb{D}(\tilde{\mathbf{u}}_I^h)\|_{L^3},\mathbb{D}(\tilde{\mathbf{u}})\|_{L^3}\}$. Note that the estimate is also valid for $\nu_0(h)=0$. Estimate (4.14) also shows that the nonlinear viscous term of the Smagorinsky model can be estimated without obtaining constants depending on the Reynolds number.

The critical estimate is the estimate of the nonlinear convective terms. One starts by splitting this term as follows

$$b_s\left(\mathbf{u},\mathbf{u},\boldsymbol{\phi}^h\right)-b_s\left(\mathbf{u}^h,\mathbf{u}^h,\boldsymbol{\phi}^h\right)=b_s\left(\mathbf{u},\boldsymbol{\eta},\boldsymbol{\phi}^h\right)+b_s\left(\boldsymbol{\eta},\mathbf{u}^h,\boldsymbol{\phi}^h\right)-b_s\left(\boldsymbol{\phi}^h,\mathbf{u}^h,\boldsymbol{\phi}^h\right).$$

The first two terms are estimated the same way by using Hölder's inequality, Korn's inequality, (3.10) and Young's inequality. One obtains

$$\begin{aligned}b_s\left(\mathbf{u},\boldsymbol{\eta},\boldsymbol{\phi}^h\right) &\leqslant \frac{1}{2}\left(\|\mathbf{u}\|_{L^6}\|\nabla\boldsymbol{\eta}\|_{L^3}\left\|\boldsymbol{\phi}^h\right\|_{L^2}+\left\|\nabla\boldsymbol{\phi}^h\right\|_{L^3}\|\mathbf{u}\|_{L^2}\|\boldsymbol{\eta}\|_{L^6}\right)\\ &\leqslant C\left(\|\mathbf{u}\|_{L^6}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3}\left\|\boldsymbol{\phi}^h\right\|_{L^2}+\left\|\mathbb{D}(\boldsymbol{\phi}^h)\right\|_{L^3}\|\mathbf{u}\|_{L^2}\|\boldsymbol{\eta}\|_{L^6}\right)\\ &\leqslant C\left(\|\mathbf{u}\|_{L^6}\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3}\left\|\boldsymbol{\phi}^h\right\|_{L^2}+\|\mathbf{u}\|_{L^2}\|\boldsymbol{\eta}\|_{L^6}\left(\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^3}+H^{-(d+6)/6}\left\|\boldsymbol{\phi}^h\right\|_{L^2}\right)\right)\\ &\leqslant C\left(\left\|\mathbb{D}(\boldsymbol{\eta})\right\|_{L^3}^2+\delta^{-1}\|\mathbf{u}\|_{L^2}^{3/2}\|\boldsymbol{\eta}\|_{L^6}^{3/2}+H^{-(d+6)/3}\|\boldsymbol{\eta}\|_{L^6}^2\right)+\frac{C_1\delta^2}{6}\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^3}^3\\ &\quad+\left(\|\mathbf{u}\|_{L^6}^2+\|\mathbf{u}\|_{L^2}^2\right)\left\|\boldsymbol{\phi}^h\right\|_{L^2}^2 \end{aligned}\tag{4.15}$$

and

$$\begin{aligned}b_s\left(\boldsymbol{\eta},\mathbf{u}^h,\boldsymbol{\phi}^h\right) &\leqslant C\left(\|\boldsymbol{\eta}\|_{L^6}^2+\delta^{-1}\left\|\mathbf{u}^h\right\|_{L^2}^{3/2}\|\boldsymbol{\eta}\|_{L^6}^{3/2}+H^{-(d+6)/3}\|\boldsymbol{\eta}\|_{L^6}^2\right)+\frac{C_1\delta^2}{6}\left\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\right\|_{L^3}^3\\ &\quad+\left(\left\|\nabla\mathbf{u}^h\right\|_{L^3}^2+\left\|\mathbf{u}^h\right\|_{L^2}^2\right)\left\|\boldsymbol{\phi}^h\right\|_{L^2}^2. \end{aligned}\tag{4.16}$$

Integration by parts and Hölder's inequality give

$$b_s(\boldsymbol{\phi}^h, \mathbf{u}^h, \boldsymbol{\phi}^h) = \big((\boldsymbol{\phi}^h \cdot \nabla)\mathbf{u}^h, \boldsymbol{\phi}^h\big) + \frac{1}{2}\big(\nabla \cdot \boldsymbol{\phi}^h, \boldsymbol{\phi}^h \cdot \mathbf{u}^h\big)$$
$$\leqslant \|\boldsymbol{\phi}^h\|_{L^3}^2 \|\nabla \mathbf{u}^h\|_{L^3} + \frac{1}{2}\big(\nabla \cdot \boldsymbol{\phi}^h, \boldsymbol{\phi}^h \cdot \mathbf{u}^h\big). \tag{4.17}$$

For estimating the first term in (4.17), the imbedding $W^{1/2,2}(\Omega) \hookrightarrow L^3(\Omega)$ and the interpolation estimate of $W^{1/2,2}(\Omega)$ between $L^2(\Omega)$ and $W^{1,2}(\Omega)$ are used, see [1]

$$\|\boldsymbol{\phi}^h\|_{L^3}^2 \|\nabla \mathbf{u}^h\|_{L^3} \leqslant C\|\nabla \mathbf{u}^h\|_{L^3} \|\boldsymbol{\phi}^h\|_{W^{1,2}} \|\boldsymbol{\phi}^h\|_{L^2}.$$

Applying in addition Poincaré's inequality, Korn's inequality, the triangle inequality, the inverse finite element inequality for the space V^H, the L^2-stability (3.4) and Young's inequality gives

$$\|\boldsymbol{\phi}^h\|_{L^3}^2 \|\nabla \mathbf{u}^h\|_{L^3} \leqslant C\big(\|\nabla \mathbf{u}^h\|_{L^3} \|\mathbb{D}(\bar{\boldsymbol{\phi}}^h)\|_{L^2} \|\boldsymbol{\phi}^h\|_{L^2} + \|\mathbb{D}(\mathbf{u}^h)\|_{L^3} \|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\|_{L^2} \|\boldsymbol{\phi}^h\|_{L^2}\big)$$
$$\leqslant \frac{\nu_0(h)}{8}\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\|_{L^2}^2 + C\left(H^{-1}\|\nabla \mathbf{u}^h\|_{L^3} + \frac{\|\nabla \mathbf{u}^h\|_{L^3}^2}{\nu_0(h)}\right)\|\boldsymbol{\phi}^h\|_{L^2}^2. \tag{4.18}$$

For estimating the second term in (4.17), $\boldsymbol{\phi}^h$ is split into the large scale and the resolved small scale part. Applying Hölder's inequality, the estimate of the norm of the divergence by the same norm of the deformation tensor and (3.9) gives

$$\frac{1}{2}\big(\nabla \cdot \boldsymbol{\phi}^h, \boldsymbol{\phi}^h \cdot \mathbf{u}^h\big) \leqslant \frac{1}{2}\big(\nabla \cdot \bar{\boldsymbol{\phi}}^h, \boldsymbol{\phi}^h \cdot \mathbf{u}^h\big) + \frac{1}{2}\big(\nabla \cdot \tilde{\boldsymbol{\phi}}^h, \boldsymbol{\phi}^h \cdot \mathbf{u}^h\big)$$
$$\leqslant \frac{1}{2}\|\nabla \cdot \bar{\boldsymbol{\phi}}^h\|_{L^3}\|\mathbf{u}^h\|_{L^6}\|\boldsymbol{\phi}^h\|_{L^2} + \frac{1}{2}\|\nabla \cdot \tilde{\boldsymbol{\phi}}^h\|_{L^2}\|\mathbf{u}^h\|_{L^t}\|\boldsymbol{\phi}^h\|_{L^s}$$
$$\leqslant C\|\mathbb{D}(\bar{\boldsymbol{\phi}}^h)\|_{L^3}\|\mathbf{u}^h\|_{L^6}\|\boldsymbol{\phi}^h\|_{L^2} + C\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\|_{L^2}\|\mathbf{u}^h\|_{L^t}\|\boldsymbol{\phi}^h\|_{L^s}$$
$$\leqslant CH^{-(d+6)/6}\|\mathbf{u}^h\|_{L^6}\|\boldsymbol{\phi}^h\|_{L^2}^2 + C\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\|_{L^2}\|\mathbf{u}^h\|_{L^t}\|\boldsymbol{\phi}^h\|_{L^s}$$

with $1/s + 1/t = 1/2$. In virtue of the Sobolev imbedding theorem $W^{1,3}(\Omega) \hookrightarrow L^t(\Omega)$, with $t \in [1, \infty]$ if $d = 2$, and $t \in [1, \infty)$ if $d = 3$, we set $s = 2$ for $d = 2$, and $s = 2 + \varepsilon_s$ with $\varepsilon_s > 0$, for $d = 3$. With (3.7), one obtains

$$\|\boldsymbol{\phi}^h\|_{L^s} \leqslant Ch^{-d\varepsilon_s/(4+2\varepsilon_s)}\|\boldsymbol{\phi}^h\|_{L^2} =: Ch^{-\frac{\varepsilon}{2}}\|\boldsymbol{\phi}^h\|_{L^2}$$

with $\varepsilon = 0$ if $d = 2$ and $\varepsilon > 0$ if $d = 3$. Young's inequality gives

$$\frac{1}{2}\big(\nabla \cdot \boldsymbol{\phi}^h, \boldsymbol{\phi}^h \cdot \mathbf{u}^h\big) \leqslant C\left(H^{-(d+6)/6}\|\mathbf{u}^h\|_{L^6} + \frac{h^{-\varepsilon}}{\nu_0(h)}\|\nabla \mathbf{u}^h\|_{L^3}^2\right)\|\boldsymbol{\phi}^h\|_{L^2}^2 + \frac{\nu_0(h)}{8}\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\|_{L^2}^2. \tag{4.19}$$

Remark 4.2. The error estimate for the Smagorinsky LES model in [29] does not possess a term with the factor $h^{-\varepsilon}$. Instead, the discrete Smagorinsky model has an additional stabilization term $\alpha(\nabla \cdot \mathbf{u}^h, \nabla \cdot \mathbf{v}^h)$ (grad–div stabilization [5]) with $\alpha > 0$ and this term is used to estimate $(\nabla \cdot \boldsymbol{\phi}^h, \boldsymbol{\phi}^h \cdot \mathbf{u}^h)$. Of course, the introduction of a grad–div stabilization into (2.6) would allow to perform a similar estimate. The absence of the grad–div stabilization results apparently in constants depending on (arbitrarily small) negative powers of h for $d = 3$, see also the discussion of this topic in [22].

Inserting the estimates (4.11)–(4.16) (4.18) and (4.19) into (4.10) leads to

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\phi}^h\|_{L^2}^2 + \nu\|\mathbb{D}(\boldsymbol{\phi}^h)\|_{L^2}^2 + \frac{\nu_0(h)}{8}\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\|_{L^2}^2 + \frac{C_1\delta^2}{6}\|\mathbb{D}(\tilde{\boldsymbol{\phi}}^h)\|_{L^3}^3$$
$$\leqslant C\Big(H^{-(d+6)/3}\|\boldsymbol{\eta}_t\|_{W^{-1,3/2}}^2 + \delta^{-1}\|\boldsymbol{\eta}_t\|_{W^{-1,3/2}}^{3/2} + \nu\|\mathbb{D}(\boldsymbol{\eta})\|_{L^2}^2 + \nu_0(h)\|\mathbb{D}(\tilde{\boldsymbol{\eta}})\|_{L^2}^2 + \tilde{C}_L^{3/2}\delta^2\|\mathbb{D}(\tilde{\boldsymbol{\eta}})\|_{L^3}^{3/2}$$
$$+ \|\mathbb{D}(\boldsymbol{\eta})\|_{L^3}^2 + \delta^{-1}\big(\|\mathbf{u}\|_{L^2}^{3/2} + \|\mathbf{u}^h\|_{L^2}^{3/2}\big)\|\boldsymbol{\eta}\|_{L^6}^{3/2} + \big(1 + H^{-(d+6)/3}\big)\|\boldsymbol{\eta}\|_{L^6}^2 + H^{-(d+6)/3}\|p - q^h\|_{L^{3/2}}^2$$
$$+ \delta^{-1}\|p - q^h\|_{L^{3/2}}^{3/2} + \nu_0(h)\|\mathbb{D}(\tilde{\mathbf{u}})\|_{L^2}^2 + \delta^2\|\mathbb{D}(\tilde{\mathbf{u}})\|_{L^3}^3\Big)$$

$$+\left(\frac{1}{2}+\|\mathbf{u}\|_{L^6}^2+\|\mathbf{u}\|_{L^2}^2+\|\mathbf{u}^h\|_{L^2}^2+CH^{-1}\|\nabla\mathbf{u}^h\|_{L^3}+\left(1+C\frac{1+h^{-\varepsilon}}{\nu_0(h)}\right)\|\nabla\mathbf{u}^h\|_{L^3}^2 + CH^{-(d+6)/6}\|\mathbf{u}^h\|_{L^6}\right)\|\boldsymbol{\phi}^h\|_{L^2}^2 \tag{4.20}$$

for all $q^h \in Q^h$.

In order to apply Gronwall's lemma to (4.20), the $L^1(0,T)$-regularity of the appearing terms has to be studied. Consider first the critical terms on the right-hand side. Let $t \in (0,T]$ be arbitrary.

From the stability estimates in Lemmas 4.1 and 4.2 it follows that $\|\mathbf{u}\|_{L^2} \in L^\infty(0,T)$ and $\|\mathbf{u}^h\|_{L^2} \in L^\infty(0,T)$. Hence

$$\int_0^t \left(\|\mathbf{u}\|_{L^2}^{3/2}+\|\mathbf{u}^h\|_{L^2}^{3/2}\right)\|\boldsymbol{\eta}\|_{L^6}^{3/2}\,d\tau \leqslant \left(\|\mathbf{u}\|_{L^\infty(0,T;L^2)}^{3/2}+\|\mathbf{u}^h\|_{L^\infty(0,T;L^2)}^{3/2}\right)\|\boldsymbol{\eta}\|_{L^{3/2}(0,T;L^6)}^{3/2} \leqslant C\|\boldsymbol{\eta}\|_{L^{3/2}(0,T;L^6)}^{3/2}.$$

Also note that, since the Lipschitz continuity coefficient $\tilde{C}_L$ is given by $\tilde{C}_L = \max\{\|\mathbb{D}(\tilde{\mathbf{u}}_I^h)\|_{L^3}, \mathbb{D}(\tilde{\mathbf{u}})\|_{L^3}\}$, by the Cauchy–Schwarz inequality, (4.3), (4.6) and (4.8), the following term is bounded

$$\int_0^t \max\left\{\|\mathbb{D}(\tilde{\mathbf{u}}_I^h)\|_{L^3}^{3/2}, \|\mathbb{D}(\tilde{\mathbf{u}})\|_{L^3}^{3/2}\right\}\|\mathbb{D}(\tilde{\boldsymbol{\eta}})\|_{L^3}^{3/2}\,d\tau$$

$$\leqslant \max\left\{\|\mathbb{D}(\tilde{\mathbf{u}}_I^h)\|_{L^3(0,t;L^3)}^{3/2}, \|\mathbb{D}(\tilde{\mathbf{u}})\|_{L^3(0,t;L^3)}^{3/2}\right\}\|\mathbb{D}(\tilde{\boldsymbol{\eta}})\|_{L^3(0,t;L^3)}^{3/2} < \infty.$$

Sobolev imbeddings, Poincaré's and Korn's inequality imply that $\|\mathbf{u}\|_{L^6}^2$, $\|\mathbf{u}^h\|_{L^6}$ and $\|\boldsymbol{\eta}\|_{L^6}^2$ are in $L^1(0,T)$. In addition, Korn's inequality and the stability estimate imply that $\|\nabla\mathbf{u}^h\|_{L^3}^2 \in L^1(0,T)$. The $L^1(0,T)$-regularity of the additional terms is a direct consequence of (4.1)–(4.6) and (4.8). Hence, by setting

$$A(t) = \frac{1}{2}+\|\mathbf{u}\|_{L^6}^2+\|\mathbf{u}\|_{L^2}^2+\|\mathbf{u}^h\|_{L^2}^2+\left(1+C\frac{1+h^{-\varepsilon}}{\nu_0(h)}\right)\|\nabla\mathbf{u}^h\|_{L^3}^2+CH^{-1}\|\nabla\mathbf{u}^h\|_{L^3} + CH^{-(d+6)/6}\|\mathbf{u}^h\|_{L^6}, \tag{4.21}$$

with $t \in [0,T]$, it follows that $A(t) \in L^1(0,T)$. Applying Gronwall's lemma and the triangle inequality to (4.20) proves the following theorem.

Theorem 4.1. *Let* $(\mathbf{u}, p)$ *be the solution of* (2.1) *and let* $\mathbf{u}^h$ *be the solution of* (2.6)*. Suppose* (4.1)*,* (4.2) *hold, and let* $A(t)$ *be given in* (4.21)*. Then there exists a constant* C^* *independent of* ν *such that*

$$\|A(t)\|_{L^1(0,T)} \leqslant C^*(\nu_0(h), H, h).$$

Moreover the error $\mathbf{u}-\mathbf{u}^h$ *satisfies for* $0 \leqslant T < \infty$:

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}^h\|_{L^\infty(0,T;L^2)}^2+\nu\|\mathbb{D}(\mathbf{u}-\mathbf{u}^h)\|_{L^2(0,T;L^2)}^2+\nu_0(h)\|\mathbb{D}(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}^h)\|_{L^2(0,T;L^2)}^2+C_1\delta^2\|\mathbb{D}(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}^h)\|_{L^3(0,T;L^3)}^3\\
&\quad\leqslant C\exp\left(C^*(\nu_0(h),H,h)\right)\|(\mathbf{u}-\mathbf{u}^h)(\mathbf{x},0)\|_{L^2}^2\\
&\qquad + C\inf_{\substack{\mathbf{u}_I^h\in V^h\\ q_I^h\in Q^h}}\Big[\|\mathbf{u}-\mathbf{u}_I^h\|_{L^\infty(0,T;L^2)}^2+\nu\|\mathbb{D}(\mathbf{u}-\mathbf{u}_I^h)\|_{L^2(0,T;L^2)}^2+\nu_0(h)\|\mathbb{D}(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_I^h)\|_{L^2(0,T;L^2)}^2\\
&\qquad + \delta^2\|\mathbb{D}(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_I^h)\|_{L^3(0,T;L^3)}^3\\
&\qquad + \exp\left(C^*(\nu_0(h),H,h)\right)\Big(\|(\mathbf{u}-\mathbf{u}_I^h)(\mathbf{x},0)\|_{L^2}^2+H^{-(d+6)/3}\|\mathbf{u}_t-(\mathbf{u}_I^h)_t\|_{L^2(0,T;W^{-1,3/2})}^2\\
&\qquad + \delta^{-1}\|\mathbf{u}_t-(\mathbf{u}_I^h)_t\|_{L^{3/2}(0,T;W^{-1,3/2})}^{3/2}+\nu\|\mathbb{D}(\mathbf{u}-\mathbf{u}_I^h)\|_{L^2(0,T;L^2)}^2+\nu_0(h)\|\mathbb{D}(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_I^h)\|_{L^2(0,T;L^2)}^2\\
&\qquad + \delta^2\max\left\{\|\mathbb{D}(\tilde{\mathbf{u}}_I^h)\|_{L^3(0,t;L^3)}^{3/2}, \|\mathbb{D}(\tilde{\mathbf{u}})\|_{L^3(0,t;L^3)}^{3/2}\right\}\|\mathbb{D}(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_I^h)\|_{L^{3/2}(0,T;L^3)}^{3/2}+\|\mathbb{D}(\mathbf{u}-\mathbf{u}_I^h)\|_{L^2(0,T;L^3)}^2
\end{aligned}$$

$$+ \delta^2 \|\mathbb{D}(\tilde{\mathbf{u}})\|^3_{L^3(0,T;L^3)} + \nu_0(h)\|\mathbb{D}(\tilde{\mathbf{u}})\|^2_{L^2(0,T;L^2)} + \delta^{-1}\big[\|\mathbf{u}(0)\|^2_{L^2} + \nu^{-1}\|\mathbf{f}\|^2_{L^2(0,t;H^{-1})}$$
$$+ \exp(t)\big(\|\mathbf{u}^h(0)\|^2_{L^2} + \delta^{-1}\|\mathbf{f}\|^{3/2}_{L^{3/2}(0,T;W^{-1,3/2})} + H^{-(d+6)/3}\|\mathbf{f}\|^2_{L^2(0,T;W^{-1,3/2})}\big)\big]^{3/2}\|\mathbf{u} - \mathbf{u}_I^h\|^{3/2}_{L^{3/2}(0,T;L^6)}$$
$$+ \big(1 + H^{-(d+6)/3}\big)\|\mathbf{u} - \mathbf{u}_I^h\|^2_{L^2(0,T;L^6)} + H^{-(d+6)/3}\|p - q_I^h\|^2_{L^2(0,T;L^{3/2})} + \delta^{-1}\|p - q_I^h\|^{3/2}_{L^{3/2}(0,T;L^{3/2})}\big)\Big].$$

The right-hand side of the error estimate depends on negative powers of the viscosity (hence on positive powers of the Reynolds number). However, this dependency is inevitable since the error to the solution of the Navier–Stokes equations is considered and the stability of these equations appears naturally in the final estimate. The influence on ν can be seen directly in the term $\nu^{-1}\|\mathbf{f}\|^2_{L^2(0,t;H^{-1})}$ and it enters indirectly in the terms $\|\mathbf{u}\|^2_{L^6}$, $\|\mathbf{u}\|^2_{L^2}$ in (4.21).

Apart from this dependency, the right-hand side of the final estimate depends on parameters of the method: $\nu_0(h)$, H and δ. In addition, there is a dependency on h in $C^*(\nu_0(h), H, h)$, which is, however, arbitrarily small. As mentioned in Remark 4.2, the inclusion of an appropriate grad–div stabilization would remove this dependency. Grad–div stabilization terms are used for instance in the bubble VMS methods of [13].

If $\nu_0(h) \to 0$, the estimate of term (4.17) has to be modified. This is easily possible using Hölder's and Young's inequality. It leads, however, to another term in the final estimate depending on ν. In the case $\nu_0(h) > 0$ but $\delta \to 0$, the estimates in Lemma 4.2 and in (4.11) can be changed such that the δ^{-1}-terms in the final estimate are replaced by $\nu_0(h)^{-1}$-terms.

There are two terms in the final estimate which do not involve factors with approximation errors: $\delta^2\|\mathbb{D}(\tilde{\mathbf{u}})\|^3_{L^3(0,T;L^3)}$ and $\nu_0(h)\|\mathbb{D}(\tilde{\mathbf{u}})\|^2_{L^2(0,T;L^2)}$. The wish that these terms do not spoil the error estimates imposes conditions on the choice of $\nu_0(h)$ and δ. The specification of these choices for a concrete example requires the knowledge of the interpolation errors of the other terms, which in turn requires knowledge on the regularity of the solution and the knowledge of the used finite element spaces. In practice, the choices $\delta = Ch$, $C \in [1, 2]$, and $\nu_0(h) = 0$, see Remark 2.1 and Section 5, are standard.

5. Numerical illustration

The numerical tests consider the projection-based VMS method with Smagorinsky-type turbulent viscosity ν_T. The stability of the method with respect to the size of the viscosity ν and the convergence with respect to the mesh width will be studied exemplarily at the Beltrami flow problem. The projection-based VMS method used in these studies defines the scale separation in a somewhat different way than the method analyzed in Section 4, see Remark 3.1. The used FEVMS is the same method which was applied in the turbulent flow simulations in [25,28,31] and for which finite element error estimates in the case of constant turbulent viscosity were derived in [26]. Based on the available computational and analytical results, we think that the two slightly different definitions of the scale separation will not affect such fundamental properties of projection-based FEVMS like stability and convergence.

The fundamental difficulty in convergence studies for the incompressible Navier–Stokes equations is that analytical expressions for the solution are generally not known for realistic flows. There are only very few academic examples which have at least some physical meaning. One of them is the so-called Beltrami flow which was used e.g. in [12]. The Beltrami flow is defined in $\Omega = (1, 1)^3$ and the prescribed solution is

$$\mathbf{u} = -\alpha \begin{pmatrix} e^{\alpha x}\sin(\alpha y + \beta z) + e^{\alpha z}\cos(\alpha x + \beta y) \\ e^{\alpha y}\sin(\alpha z + \beta x) + e^{\alpha x}\cos(\alpha y + \beta z) \\ e^{\alpha z}\sin(\alpha x + \beta y) + e^{\alpha y}\cos(\alpha z + \beta x) \end{pmatrix} e^{-\nu\beta^2 t},$$

$$p = -\frac{\alpha^2}{2}\big[e^{2\alpha x} + e^{2\alpha y} + e^{2\alpha z} + 2\sin(\alpha x + \beta y)\cos(\alpha z + \beta x)e^{\alpha(y+z)} + 2\sin(\alpha y + \beta z)\cos(\alpha x + \beta y)e^{\alpha(z+x)}$$
$$+ 2\sin(\alpha z + \beta x)\cos(\alpha y + \beta z)e^{\alpha(x+y)}\big]e^{-2\nu\beta^2 t},$$

where $\mathbf{x} = (x, y, z)^T$ and α and β are parameters. They were set to $\alpha = \pi/4$, $\beta = \pi/2$, see [9] for this proposal. The characteristic feature of this flow is a series of counter-rotating vortices intersecting one another at oblique angles. The time-dependent factors in the solution show the exponential decay in time of the initial configuration.

The projection-based VMS method was discretized in time with the Crank–Nicolson scheme and the equidistant time step 0.001. We checked with smaller time steps that the discretization error in space dominates the discretization

Table 1
Information on the grids and the number of degrees of freedom

k	Cells	X^h	Q^h	$L^H = P_0$	$L^H = P_1^{\text{disc}}$
2	64	2187	256	384	1536
3	512	14 739	2048	3072	12 288
4	4096	107 811	16 384	24 576	98 304
5	32 768	823 875	131 072	196 608	786 432

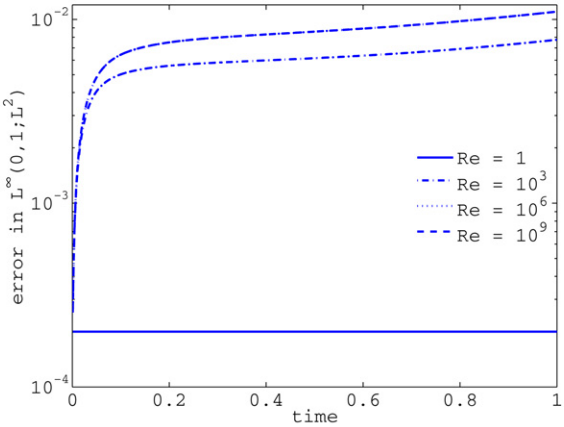

Fig. 1. Stability of the projection-based VMS method with respect to the size of the viscosity, $h = 2^{-4}$, $\nu = Re^{-1}$. The curves for $\nu = 10^{-6}$ and $\nu = 10^{-9}$ are on top of each other.

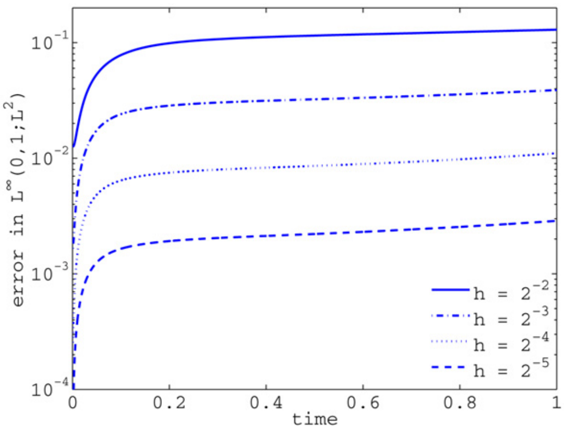

Fig. 2. Convergence of the projection-based VMS method, $\nu = 10^{-9}$.

error in time with this choice. The final time was set to $T = 1$. The velocity was approximated with the Q_2 finite element and the pressure with the P_1^{disc} finite element. This pair of finite element spaces fulfills the inf–sup condition (2.4), see [34], and it is one of the best performing pairs of finite elements in incompressible flow computations [14,23,24]. The Smagorinsky-type turbulent viscosity model was chosen as in (2.7) with $C_S = 0.01$, δ the diameter of the mesh cells and $\nu_0(h) = 0$, see also Remark 2.1. The projection-based VMS method from [25,28,31] requires the choice of a tensor-valued space L^H for the definition of the projection, see Remark 3.1. Numerical studies were performed for $L^H = P_0$ and $L^H = P_1^{\text{disc}}$. Since the qualitative results were the same for both spaces, for shortness only the results obtained with $L^H = P_0$ will be presented. The computational grids consisted of cubes with edge length 2^{-k}, $k \in \{2, 3, 4, 5\}$. Information on the number of grid cells and on the number of degrees of freedom (including

Dirichlet nodes) are given in Table 1. Since L^H is a space consisting of symmetric tensors, only six components of these tensors have to be stored. The computations were performed with the code MooNMD [30].

Results are presented for $\|\mathbf{u}-\mathbf{u}^h\|_{L^\infty(0,T;L^2)}$. Fig. 1 demonstrates the stability of the projection-based VMS method with respect to the size of the viscosity. The errors in $L^\infty(0,T;L^2)$ are nearly the same for $\nu = 10^{-6}$ and $\nu = 10^{-9}$. At the final time, the difference between these two errors is around $5 \cdot 10^{-6}$. The numerical results show that for sufficiently small viscosity ν, the contribution of the turbulent viscosity dominates the overall viscosity in the projection-based VMS method. The application of the Smagorinsky model to the subspace of the resolved small scales was sufficient to perform stable simulations. The results on the other grid levels are similar to those presented in Fig. 1.

Fig. 2 shows the convergence of the projection-based VMS method in the case $\nu = 10^{-9}$. The error reduction between subsequent levels tends to the factor of four (it is 3.77 between $h = 2^{-4}$ and $h = 2^{-5}$ at the final time). Thus, if the discretization error in time is kept sufficiently small, second order convergence can be expected. The results obtained with other values of the viscosity are qualitatively the same.

Acknowledgment

The research of A. Kindl was supported by the Deutsche Forschungsgemeinschaft (DFG) by grant No. Jo 329/7-1.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] L.C. Berselli, C.R. Grisanti, V. John, Analysis of commutation errors for functions with low regularity, J. Comput. Appl. Math. 206 (2007) 1027–1045.
[3] L.C. Berselli, T. Iliescu, W.J. Layton, Mathematics of Large Eddy Simulation of Turbulent Flows, Springer-Verlag, 2006.
[4] L.C. Berselli, V. John, Asymptotic behavior of commutation errors and the divergence of the Reynolds stress tensor near the wall in the turbulent channel flow, Math. Methods Appl. Sci. 29 (2006) 1709–1719.
[5] M. Braack, E. Burman, V. John, G. Lube, Stabilized finite element methods for the generalized Oseen problem, Comput. Methods Appl. Mech. Engrg. 196 (2007) 853–866.
[6] P.G. Ciarlet, Basic error estimates for elliptic problems, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis II, North-Holland, Amsterdam, New York, Oxford, Tokyo, 1991, pp. 19–351.
[7] S.S. Collis, Monitoring unresolved scales in multiscale turbulence modeling, Phys. Fluids 13 (2001) 1800–1806.
[8] A. Dunca, V. John, W.J. Layton, The commutation error of the space averaged Navier–Stokes equations on a bounded domain, in: J.G. Heywood, G.P. Galdi, R. Rannacher (Eds.), Contributions to Current Challenges in Mathematical Fluid Mechanics, Birkhäuser, 2004, pp. 53–78.
[9] C.R. Ethier, D.A. Steinmann, Exact fully 3D Navier–Stokes solution for benchmarking, Internat. J. Numer. Methods Fluids 19 (1994) 369–375.
[10] G.P. Galdi, An introduction to the Navier–Stokes initial-boundary value problem, in: G.P. Galdi, J.G. Heywood, R. Rannacher (Eds.), Fundamental Directions in Mathematical Fluid Dynamics, Birkhäuser, 2000, pp. 1–70.
[11] V. Gravemeier, Scale-separting operators for variational multiscale large eddy simulation of turbulent flows, J. Comput. Phys. 212 (2006) 400–435.
[12] V. Gravemeier, W.A. Wall, E. Ramm, A three-level finite element method for the instationary incompressible Navier–Stokes equation, Comput. Methods Appl. Mech. Engrg. 193 (2004) 1323–1366.
[13] V. Gravemeier, W.A. Wall, E. Ramm, Large eddy simulation of turbulent incompressible flows by a three-level finite element method, Internat. J. Numer. Methods Fluids 48 (2005) 1067–1099.
[14] P.M. Gresho, R.L. Sani, Incompressible Flow and the Finite Element Method, Wiley, Chichester, 2000.
[15] J.-L. Guermond, Stabilization of Galerkin approximations of transport equations by subgrid modeling, M2AN Math. Model. Numer. Anal. 33 (1999) 1293–1316.
[16] T.J.R. Hughes, L. Mazzei, K.E. Jansen, Large eddy simulation and the variational multiscale method, Comput. Vis. Sci. 3 (2000) 47–59.
[17] T.J.R. Hughes, Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid-scale models, bubbles and the origin of stabilized methods, Comput. Methods Appl. Mech. Engrg. 127 (1995) 387–401.
[18] T.J.R. Hughes, L. Mazzei, A.A. Oberai, A.A. Wray, The multiscale formulation of large eddy simulation: Decay of homogeneous isotropic turbulence, Phys. Fluids 13 (2001) 505–512.
[19] T.J.R. Hughes, A.A. Oberai, L. Mazzei, Large eddy simulation of turbulent channel flows by the variational multiscale method, Phys. Fluids 13 (2001) 1784–1799.
[20] T. Iliescu, V. John, W.J. Layton, Convergence of finite element approximations of large eddy motion, Numer. Methods Partial Differential Equations 18 (2002) 689–710.
[21] K.E. Jansen, A.E. Tejada-Martinez, An evaluation of the variational multiscale model for large eddy simulation while using a hierarchical basis, AIAA Paper 2002-0283, 2002.

[22] V. John, Large Eddy Simulation of Turbulent Incompressible Flows, Analytical and Numerical Results for a Class of LES Models, Lect. Notes Comput. Sci. Eng., vol. 34, Springer-Verlag, Berlin, Heidelberg, New York, 2004.
[23] V. John, Reference values for drag and lift of a two-dimensional time dependent flow around a cylinder, Internat. J. Numer. Methods Fluids 44 (2004) 777–788.
[24] V. John, On the efficiency of linearization schemes and coupled multigrid methods in the simulation of a 3d flow around a cylinder, Internat. J. Numer. Methods Fluids 50 (2006) 845–862.
[25] V. John, S. Kaya, A finite element variational multiscale method for the Navier–Stokes equations, SIAM J. Sci. Comput. 26 (2005) 1485–1503.
[26] V. John, S. Kaya, Finite element error analysis of a variational multiscale method for the Navier–Stokes equations, Adv. Comput. Math. 28 (2008) 43–61.
[27] V. John, A. Kindl, On finite element variational multiscale methods for turbulent flow simulations, preprint, Universität des Saarlandes, FR 6.1 Mathematik, 2007.
[28] V. John, A. Kindl, Variants of projection-based finite element variational multiscale methods for the simulation of turbulent flows, Internat. J. Numer. Methods Fluids 56 (2008) 1321–1328.
[29] V. John, W.J. Layton, Analysis of numerical errors in large eddy simulation, SIAM J. Numer. Anal. 40 (2002) 995–1020.
[30] V. John, G. Matthies, MooNMD—a program package based on mapped finite element methods, Comput. Vis. Sci. 6 (2004) 163–170.
[31] V. John, M. Roland, Simulations of the turbulent channel flow at $Re_\tau = 180$ with projection-based finite element variational multiscale methods, Internat. J. Numer. Methods Fluids 55 (2007) 407–429.
[32] O.A. Ladyzhenskaya, New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them, Proc. Steklov Inst. Math. 102 (1967) 95–118.
[33] W.J. Layton, A nonlinear, subgridscale model for incompressible viscous flow problems, SIAM J. Sci. Comput. 17 (2) (1996) 347–357.
[34] G. Matthies, L. Tobiska, The inf–sup condition for the mapped $Q_k/P_{k-1}^{\mathrm{disc}}$ element in arbitrary space dimensions, Computing 69 (2002) 119–139.
[35] U. Piomelli, E. Balaras, Wall–layer models for large eddy simulation, Annu. Rev. Fluid Mech. 34 (2002) 349–374.
[36] S.B. Pope, Turbulent Flows, Cambridge Univ. Press, 2000.
[37] S. Ramakrishnan, S.S. Collis, Turbulence control simulation using the variational multiscale method, AIAA Journal 42 (2004) 745–753.
[38] J. Serrin, The Initial Value Problem for the Navier–Stokes Equations, Univ. of Wisconsin Press, Madison, WI, 1963.
[39] H. Sohr, The Navier–Stokes Equations, an Elementary Functional Analytic Approach, Birkhäuser Advanced Texts, Birkhäuser, Basel, Boston, Berlin, 2001.
[40] F. van der Bos, B.J. Geurts, Commutator erros in the filtering approach to large eddy simulation, Phys. Fluids 17 (2005) 035108.