A nonlinear local projection stabilization for convection-diffusion-reaction equations

Gabriel R. Barrenechea, Volker John, and Petr Knobloch

Abstract We propose a new local projection stabilization (LPS) finite element method for convection-diffusion-reaction equations. The discretization contains a crosswind diffusion term which depends on the unknown discrete solution in a nonlinear way. Consequently, the resulting method is nonlinear. Solvability of the nonlinear problem is established and an a priori error estimate in the LPS norm is proved. Numerical results show that the nonlinear crosswind diffusion term leads to a reduction of spurious oscillations.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be a bounded polygonal (polyhedral) domain with a Lipschitz-continuous boundary $\partial\Omega$ and let us consider the steady-state convection-diffusion-reaction equation

$$-\varepsilon\,\Delta u + \mathbf{b}\cdot\nabla u + c\,u = f \quad \text{in } \Omega\,, \qquad\qquad u = u_b \quad \text{on } \partial\Omega\,. \tag{1}$$

It is assumed that ε is a positive constant and $\mathbf{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, and $u_b \in H^{1/2}(\partial\Omega)$ are given functions satisfying

Gabriel R. Barrenechea
University of Strathclyde, Department of Mathematics and Statistics, 26 Richmond Street, Glasgow G1 1XH, Scotland, e-mail: gabriel.barrenechea@strath.ac.uk

Volker John
Weierstrass Institute for Applied Analysis and Stochastics (WIAS), Mohrenstr. 39, 10117 Berlin, Germany and Free University of Berlin, Department of Mathematics and Computer Science, Arnimallee 6, 14195 Berlin, Germany, e-mail: volker.john@wias-berlin.de

Petr Knobloch
Charles University, Faculty of Mathematics and Physics, Department of Numerical Mathematics, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: knobloch@karlin.mff.cuni.cz

$$\sigma := c - \frac{1}{2}\nabla\cdot\mathbf{b} \geq \sigma_0 > 0 \quad \text{in } \Omega\,,$$

where σ_0 is a constant. Then the boundary value problem (1) has a unique solution in $H^1(\Omega)$.

The numerical solution of (1) is still a challenge if convection dominates diffusion. In the framework of the finite element method, the common approach is to apply a stabilized method, see [5] for a review. Linear stabilized methods typically provide approximate solutions that possess spurious oscillations in layer regions. These oscillations can be suppressed without smearing the layers significantly by adding an additional artificial diffusion term depending on the approximate solution in a nonlinear way, see [2] for a review of various approaches of this type that we call spurious oscillations at layers diminishing (SOLD) methods.

Here we concentrate on local projection stabilizations (LPS) [1, 3, 4]. In comparison with residual-based methods, the linear LPS has several advantages. In particular, it does not contain second order derivatives, which may be costly to implement, and if applied to systems of PDEs, it does not lead to additional couplings between various unknowns. To suppress oscillations in layer regions, we design a new nonlinear stabilization term inspired by both the linear LPS and the above-mentioned nonlinear SOLD methods. Since we assume that the linear LPS adds enough artificial diffusion in the streamline direction, we introduce only crosswind diffusion through the nonlinear term. To preserve the above-mentioned advantages of the LPS, the residual usually appearing in SOLD terms is replaced by a fluctuation of the crosswind derivative of the approximate solution. This makes sense since the additional stabilization should be added in regions where oscillations in the crosswind direction appear. For the resulting nonlinear method, we prove the existence of a solution, without any restriction on the multiplicative factor in the nonlinear term. Furthermore, we establish an a priori error estimate with respect to the standard LPS norm. The properties of the new method are illustrated by numerical results. Let us mention that such results cannot be obtained using a linear crosswind-diffusion term since then a reduction of spurious oscillations would be possible only at the price of a considerable smearing of the layers.

The plan of the paper is as follows. Section 2 will summarize the main abstract hypothesis imposed on the different partitions of Ω and the finite element spaces considered. Section 3 presents the method whose well-posedness is analyzed in Section 4. An a priori error estimates is derived in Section 5. Finally, numerical results are presented in Section 6.

2 Assumptions

Given $h > 0$, let $W_h \subset W^{1,\infty}(\Omega)$ be a finite-dimensional space approximating the space $H^1(\Omega)$ and set $V_h = W_h \cap H^1_0(\Omega)$. Next, let $\mathscr{M}_h$ be a set consisting of a finite number of open subsets M of Ω such that $\overline{\Omega} = \cup_{M\in\mathscr{M}_h}\overline{M}$. It will be supposed that, for any $M \in \mathscr{M}_h$,

$$\mathrm{card}\{M' \in \mathscr{M}_h;\, M \cap M' \neq \emptyset\} \leq C\,, \tag{2}$$
$$h_M := \mathrm{diam}(M) \leq C\,h\,, \tag{3}$$
$$h_M \leq C\,h_{M'} \qquad \forall\, M' \in \mathscr{M}_h,\, M \cap M' \neq \emptyset\,. \tag{4}$$

The space W_h is assumed to satisfy the inverse inequality $|v_h|_{1,M} \leq C\,h_M^{-1}\,\|v_h\|_{0,M}$ for any $v_h \in W_h$, $M \in \mathscr{M}_h$. For any $M \in \mathscr{M}_h$, a finite-dimensional space $D_M \subset L^\infty(M)$ is introduced. It is assumed that there exists a positive constant β_{LP} independent of h such that

$$\sup_{v \in V_M} \frac{(v,q)_M}{\|v\|_{0,M}} \geq \beta_{LP}\,\|q\|_{0,M} \qquad \forall\, q \in D_M,\, M \in \mathscr{M}_h\,,$$

where $V_M = \{v_h \in V_h;\, v_h = 0 \text{ in } \Omega \setminus M\}$. Furthermore, for any $M \in \mathscr{M}_h$, a finite-dimensional space $G_M \subset L^\infty(M)$ containing the space D_M is introduced such that $(\partial v_h/\partial x_i)|_M \in G_M$ for any $v_h \in W_h$, $i = 1,\dots,d$, and it is assumed that

$$\|q\|_{0,\infty,M} \leq C\,h_M^{-\frac{d}{2}}\,\|q\|_{0,M} \qquad \forall\, q \in G_M,\, M \in \mathscr{M}_h\,. \tag{5}$$

To characterize the approximation properties of the spaces W_h and D_M, it is assumed that there exist interpolation operators $i_h \in \mathscr{L}(H^2(\Omega),W_h) \cap \mathscr{L}(H^2(\Omega) \cap H_0^1(\Omega),V_h)$ and $j_M \in \mathscr{L}(H^1(M),D_M)$, $M \in \mathscr{M}_h$, such that, for some constants $l \in \mathbb{N}$ and $C > 0$ and for any set $M \in \mathscr{M}_h$, it holds

$$|v - i_h v|_{1,M} + h_M^{-1}\,\|v - i_h v\|_{0,M} \leq C\,h_M^k\,|v|_{k+1,M} \quad \forall\, v \in H^{k+1}(M),\; k = 1,\dots,l\,, \tag{6}$$
$$\|q - j_M q\|_{0,M} \leq C\,h_M^k\,|q|_{k,M} \qquad \forall\, q \in H^k(M),\; k = 1,\dots,l\,. \tag{7}$$

We refer to [3] for examples of spaces W_h and D_M possessing the properties formulated in this section.

3 A local projection discretization

The weak form of problem (1) is: Find $u \in H^1(\Omega)$ such that $u = u_b$ on $\partial\Omega$ and

$$a(u,v) = (f,v) \qquad \forall\, v \in H_0^1(\Omega)\,, \tag{8}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ and the bilinear form a is given by

$$a(u,v) = \varepsilon\,(\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u, v) + (c\,u, v)\,.$$

For any $M \in \mathscr{M}_h$, a continuous linear projection operator π_M is introduced which maps the space $L^2(M)$ onto the space D_M. It is assumed that $\|\pi_M\|_{\mathscr{L}(L^2(M),L^2(M))} \leq C$ for any $M \in \mathscr{M}_h$. Using this operator, the fluctuation operator $\kappa_M := id - \pi_M$ is defined, where id is the identity operator on $L^2(M)$. Then, clearly

$$\|\kappa_M\|_{\mathscr{L}(L^2(M),L^2(M))} \le C \qquad \forall\, M \in \mathscr{M}_h\,. \tag{9}$$

Since κ_M vanishes on D_M, it follows from (9) and (7) that

$$\|\kappa_M\, q\|_{0,M} \le C\, h_M^k\, |q|_{k,M} \quad \forall\, q \in H^k(M),\ M \in \mathscr{M}_h,\ k = 0,\dots,l\,. \tag{10}$$

An application of κ_M to a vector-valued function means that κ_M is applied componentwise.

For any $M \in \mathscr{M}_h$, a constant $\mathbf{b}_M \in \mathbb{R}^d$ is chosen such that

$$|\mathbf{b}_M| \le \|\mathbf{b}\|_{0,\infty,M}\,, \qquad \|\mathbf{b}-\mathbf{b}_M\|_{0,\infty,M} \le C\, h_M\, |\mathbf{b}|_{1,\infty,M}\,. \tag{11}$$

A typical choice for $\mathbf{b}_M$ is the value of $\mathbf{b}$ at one point of M, or the integral mean value of $\mathbf{b}$ over M. In addition, a function $\widetilde{u}_{bh} \in W_h$ is introduced such that its trace approximates the boundary condition u_b.

We are now ready to present the finite element method to be studied: Find $u_h \in W_h$ such that $u_h - \widetilde{u}_{bh} \in V_h$ and

$$a(u_h,v_h) + s_h(u_h,v_h) + d_h(u_h;u_h,v_h) = (f,v_h) \qquad \forall\, v_h \in V_h\,, \tag{12}$$

where

$$s_h(u,v) = \sum_{M\in\mathscr{M}_h} \tau_M\, (\kappa_M(\mathbf{b}_M\cdot\nabla u), \kappa_M(\mathbf{b}_M\cdot\nabla v))_M\,,$$
$$d_h(w;u,v) = \sum_{M\in\mathscr{M}_h} (\tau_M^{\text{sold}}(w)\, \kappa_M(P_M\nabla u), \kappa_M(P_M\nabla v))_M\,,$$

$(\cdot,\cdot)_M$ is the inner product in $L^2(M)$ and $P_M : \mathbb{R}^d \to \mathbb{R}^d$ is the orthogonal projection onto the line (plane) orthogonal to $\mathbf{b}_M$. The stabilization parameters are given by

$$\tau_M = \tau_0 \min\left\{\frac{h_M}{\|\mathbf{b}\|_{0,\infty,M}}, \frac{h_M^2}{\varepsilon}\right\},$$

$$\tau_M^{\text{sold}}(u_h) = \begin{cases} \beta\, h_M\, |\mathbf{b}_M|\, \dfrac{h_M^d\, |\kappa_M(P_M\nabla u_h)|^2}{|u_h|_{1,M}^2} & \text{if } |u_h|_{1,M} \ne 0,\\ 0 & \text{if } |u_h|_{1,M} = 0, \end{cases}$$

where τ_0 and β are positive constants.

Remark 1. Using (11), (9), and $\|P_M\|_2 = 1$, one obtains

$$\|\tau_M^{\text{sold}}(v)\|_{0,1,M} \le C\, h_M^{1+d}\, \|\mathbf{b}\|_{0,\infty,M} \qquad \forall\, v \in H^1(\Omega), M \in \mathscr{M}_h\,. \tag{13}$$

In the analysis, the error will be measured using the following mesh-dependent norm

$$\|v\|_{\text{LPS}} := \left(\varepsilon\, |v|_{1,\Omega}^2 + \|\sigma^{1/2}\, v\|_{0,\Omega}^2 + s_h(v,v)\right)^{1/2}.$$

Note that integrating by parts gives

$$a(v,v)+s_h(v,v)=\|v\|_{\mathrm{LPS}}^2 \qquad \forall\, v\in H_0^1(\Omega)\,. \tag{14}$$

4 Well-posedness of the nonlinear discrete problem

This section studies the existence of solutions for the nonlinear discrete problem (12). Let us define the nonlinear operator $T_h: V_h\to V_h$ by

$$(T_h z_h,v_h)=a(z_h+\widetilde{u}_{bh},v_h)+s_h(z_h+\widetilde{u}_{bh},v_h)+d_h(z_h+\widetilde{u}_{bh};z_h+\widetilde{u}_{bh},v_h)-(f,v_h)$$

for any $z_h,v_h\in V_h$. Then $u_h\in W_h$ is a solution of (12) if and only if $u_h|_{\partial\Omega}=\widetilde{u}_{bh}|_{\partial\Omega}$ and $T_h(u_h-\widetilde{u}_{bh})=0$. Thus, our aim is to prove that the operator T_h has a zero in V_h. To this end, we shall use the following simple consequence of Brouwer's fixed-point theorem.

Lemma 1. *Let X be a finite-dimensional Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $P:X\to X$ be a continuous mapping and $K>0$ a real number such that $(Px,x)>0$ for any $x\in X$ with $\|x\|=K$. Then there exists $x\in X$ such that $\|x\|\le K$ and $Px=0$.*

Proof. See [6, p. 164, Lemma 1.4].

Theorem 1. *The problem (12) has a solution.*

Proof. In view of (14), for any $z_h\in V_h$, it holds

$$\begin{aligned}(T_h z_h,z_h)&=\|z_h\|_{\mathrm{LPS}}^2+d_h(z_h+\widetilde{u}_{bh};z_h,z_h)\\&\quad+a(\widetilde{u}_{bh},z_h)+s_h(\widetilde{u}_{bh},z_h)+d_h(z_h+\widetilde{u}_{bh};\widetilde{u}_{bh},z_h)-(f,z_h).\end{aligned}$$

According to (13), one has

$$|d_h(u;v,z)|\le C\sum_{M\in\mathscr{M}_h}h_M^{1+d}\,\|\mathbf{b}\|_{0,\infty,M}\,\|\kappa_M(P_M\nabla v)\|_{0,\infty,M}\,\|\kappa_M(P_M\nabla z)\|_{0,\infty,M}$$

for any $u,v,z\in W^{1,\infty}(\Omega)$. Thus, applying (5), (9), the equivalence of norms on finite-dimensional spaces, the Cauchy-Schwarz inequality, and the Young inequality, one deduces that

$$(T_h z_h,z_h)\ge\frac{1}{2}\,\|z_h\|_{\mathrm{LPS}}^2-C_0\,(\|\widetilde{u}_{bh}\|_{0,\Omega}^2+\|f\|_{0,\Omega}^2)\,,$$

where $C_0>0$ depends on ε, $\mathbf{b}$, c, σ_0, h, and W_h but not on z_h. Consequently,

$$(T_h z_h,z_h)\ge C_1\,\|z_h\|_{0,\Omega}^2-C_2 \qquad \forall\, z_h\in V_h\,,$$

where C_1, C_2 are positive constants. Thus, in view of Lemma 1 with any $K>\sqrt{C_2/C_1}$, the operator T_h has a zero and hence the problem (12) has a solution.

5 Error estimate

Lemma 2. *There exists an operator $\rho_h : L^2(\Omega) \to V_h$ such that, for any $v, w \in L^2(\Omega)$, the following estimates hold*

$$|(v-\rho_h v, w)| \le C \sum_{M\in\mathscr{M}_h} \|v\|_{0,M}\, \|\kappa_M w\|_{0,M}\,, \tag{15}$$

$$|\rho_h v|_{1,M}^2 + h_M^{-2}\,\|\rho_h v\|_{0,M}^2 \le C \sum_{\substack{M'\,\in\,\mathscr{M}_h,\\ M\cap M'\neq\emptyset}} h_{M'}^{-2}\,\|v\|_{0,M'}^2 \qquad \forall\, M\in\mathscr{M}_h\,. \tag{16}$$

Proof. See [4, Lemma 1].

Using the operators i_h and ρ_h, we introduce the operator $r_h \in \mathscr{L}(H^2(\Omega), W_h) \cap \mathscr{L}(H^2(\Omega)\cap H_0^1(\Omega), V_h)$ by $r_h v := i_h v + \rho_h(v - i_h v)$. To formulate the interpolation properties of r_h, it is convenient to introduce the mesh dependent norm

$$\|v\|_{1,h} = \left(\sum_{M\in\mathscr{M}_h} \{|v|_{1,M}^2 + h_M^{-2}\,\|v\|_{0,M}^2\} \right)^{1/2}.$$

Then, using (16), (2), (3), and (6), one obtains

$$\|v - r_h v\|_{1,h} \le C\,\|v - i_h v\|_{1,h} \le \tilde{C}\,h^k\,|v|_{k+1,\Omega} \qquad \forall\, v\in H^{k+1}(\Omega),\ k=1,\dots,l\,. \tag{17}$$

Lemma 3. *Let $u\in H^{k+1}(\Omega)$ for some $k\in\{1,\dots,l\}$, and let $\eta := u - r_h u$. Then, for any $v_h \in V_h\setminus\{0\}$, the following estimate holds*

$$\begin{aligned}\|\eta\|_{\mathrm{LPS}} &+ \frac{a(\eta,v_h)+s_h(\eta,v_h)-s_h(u,v_h)}{\|v_h\|_{\mathrm{LPS}}}\\ &\le C\left(\varepsilon + h\,\|\mathbf{b}\|_{0,\infty,\Omega} + h^2\,\|\sigma\|_{0,\infty,\Omega} + h^2\,|\mathbf{b}|_{1,\infty,\Omega}^2\,\sigma_0^{-1}\right)^{1/2} h^k\,|u|_{k+1,\Omega}\,. \end{aligned}\tag{18}$$

Proof. See [4].

Lemma 4. *For any $w_h\in W_h$ and $u\in H^{k+1}(\Omega)$ with $k\in\{1,\dots,l\}$, it holds*

$$d_h(w_h; r_h u, r_h u) \le C\,\|\mathbf{b}\|_{0,\infty,\Omega}\,h^{2k+1}\,|u|_{k+1,\Omega}^2\,. \tag{19}$$

Proof. First, the application of (5), (13), and (3) leads to

$$\begin{aligned} d_h(w_h; r_h u, r_h u) &\le \sum_{M\in\mathscr{M}_h} \|\tau_M^{\mathrm{sold}}(w_h)\|_{0,1,M}\,\|\kappa_M(P_M\nabla(r_h u))\|_{0,\infty,M}^2\\ &\le C\,h\,\|\mathbf{b}\|_{0,\infty,\Omega} \sum_{M\in\mathscr{M}_h} \|\kappa_M\nabla(r_h u)\|_{0,M}^2\,. \end{aligned}$$

Using (9) and (10), for $u\in H^{k+1}(\Omega)$ with $k\in\{1,\dots,l\}$ there holds

$$\begin{aligned}\|\kappa_M \nabla(r_h u)\|_{0,M} &\le \|\kappa_M \nabla u\|_{0,M} + \|\kappa_M \nabla(u - r_h u)\|_{0,M}\\ &\le C\,h_M^k\,|u|_{k+1,M} + C\,|u - r_h u|_{1,M}\,.\end{aligned}$$

Thus, (19) follows from (2), (3), and (17).

We are now in a position to prove the main result of this paper.

Theorem 2. *Let the weak solution of (1) satisfy $u \in H^{k+1}(\Omega)$ for some $k \in \{1,\dots,l\}$. Let $\widetilde{u}_b \in H^2(\Omega)$ be an extension of u_b and let $\widetilde{u}_{bh} = i_h \widetilde{u}_b$. Then the solution u_h of the local projection discretization (12) satisfies the error estimate*

$$\|u - u_h\|_{\mathrm{LPS}} \le C\left(\varepsilon + h\,\|\mathbf{b}\|_{0,\infty,\Omega} + h^2\,\|\sigma\|_{0,\infty,\Omega} + h^2\,|\mathbf{b}|^2_{1,\infty,\Omega}\,\sigma_0^{-1}\right)^{1/2} h^k\,|u|_{k+1,\Omega}\,.$$

Proof. Set $\eta := u - r_h u$ and $e_h := u_h - r_h u$. From (12) and (8), it follows that

$$\begin{aligned}&a(e_h,e_h) + s_h(e_h,e_h) + d_h(u_h;u_h,e_h)\\ &\qquad = a(u_h,e_h) + s_h(u_h,e_h) + d_h(u_h;u_h,e_h) - a(r_h u,e_h) - s_h(r_h u,e_h)\\ &\qquad = a(\eta,e_h) + s_h(\eta,e_h) - s_h(u,e_h)\,.\end{aligned}$$

Thus, in view of (14), one gets

$$\|e_h\|^2_{\mathrm{LPS}} + d_h(u_h;e_h,e_h) = a(\eta,e_h) + s_h(\eta,e_h) - s_h(u,e_h) - d_h(u_h;r_h u,e_h)\,.$$

The first three terms on the right-hand side can be estimated using (18). Applying Hölder's and Young's inequalities, one gets $d_h(u_h;r_h u,e_h) \le d_h(u_h;r_h u,r_h u) + \frac{1}{4}\,d_h(u_h;e_h,e_h)$. Therefore, using (19), one obtains

$$\begin{aligned}&\|e_h\|^2_{\mathrm{LPS}} + d_h(u_h;e_h,e_h)\\ &\qquad \le C\left(\varepsilon + h\,\|\mathbf{b}\|_{0,\infty,\Omega} + h^2\,\|\sigma\|_{0,\infty,\Omega} + h^2\,|\mathbf{b}|^2_{1,\infty,\Omega}\,\sigma_0^{-1}\right) h^{2k}\,|u|^2_{k+1,\Omega}\,.\end{aligned}$$

Finally, using the triangle inequality and the estimate (18), the statement of the theorem follows.

6 Numerical results

In this section we illustrate the properties of the method proposed in this paper by numerical results obtained for the following example.

Example 1. Solution with two interior layers. Equation (1) is considered with $\Omega = (0,1)^2$, $\varepsilon = 10^{-8}$, $\mathbf{b}(x,y) = (-y,x)^T$, $c = f = 0$, and the boundary conditions

$$u = u_b \quad \text{on } \Gamma^D\,, \qquad\qquad \frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma^N\,,$$

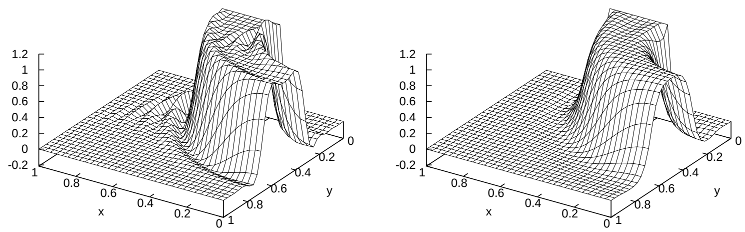

Fig. 1 LPS solutions for $\tau_0 = 0.02$, $\beta = 0$ (left) and $\tau_0 = 0.02$, $\beta = 0.05$ (right).

where $\Gamma^N = \{0\} \times (0,1)$, $\Gamma^D = \partial\Omega \setminus \overline{\Gamma^N}$, $\mathbf{n}$ is the outward pointing unit normal vector to the boundary of Ω, and

$$u_b(x,y) = \begin{cases} 1 & \text{for } (x,y) \in (1/3,2/3) \times \{0\}, \\ 0 & \text{else on } \Gamma^D. \end{cases}$$

We used a triangulation $\mathscr{T}_h$ of Ω constructed by dividing Ω into 32×32 equal squares and each square into two triangles by drawing a diagonal from bottom left to top right. Each set $M \in \mathscr{M}_h$ is the union of all triangles of $\mathscr{T}_h$ possessing a common interior vertex of $\mathscr{T}_h$. Thus the sets from $\mathscr{M}_h$ generally overlap. The space W_h consists of continuous piecewise linear functions and the spaces D_M are spaces of constant functions. Figure 1 shows that the crosswind diffusion term d_h leads to a reduction of spurious oscillations compared to the standard linear LPS method.

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