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UNIFORM BOUNDEDNESS OF NORMS OF CONVEX AND NONCONVEX PROCESSES

René Henrion[1] and Alberto Seeger[2]

[1] *Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany*
[2] *Department of Mathematics, University of Avignon, Avignon, France*

□ *The lower limit $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ of a sequence of closed convex processes $F_\nu : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is again a closed convex process. In this note, we prove the following uniform boundedness principle: if F is nonempty-valued everywhere, then there is a positive integer ν_0 such that the tail $\{F_\nu\}_{\nu\geq\nu_0}$ is "uniformly bounded" in the sense that the norms $\|F_\nu\|$ are bounded by a common constant. As shown with an example, the uniform boundedness principle is not true if one drops convexity. By way of illustration, we consider an application to the controllability analysis of differential inclusions.*

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AMS Subject Classification 34A60; 47H04; 52A20.

1. INTRODUCTION

Quite often in practice, one has to deal with sequences of graph-closed positively homogeneous multivalued maps. Recall that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be positively homogeneous if $0 \in F(0)$ and $F(\alpha x) = \alpha F(x)$ for all $\alpha > 0$ and $x \in \mathbb{R}^n$. Graph-closedness of F means that

$$\operatorname{gr} F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\}$$

is a closed set in the product space $\mathbb{R}^n \times \mathbb{R}^m$. For the sake of brevity, a graph-closed positively homogeneous multivalued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is referred to as a closed process from $\mathbb{R}^n$ to $\mathbb{R}^m$. One omits mentioning the underlying Euclidean spaces when everything is clear from the context.

Address correspondence to Prof. René Henrion, Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany; E-mail: henrion@wias-berlin.de

Observe that the graph of a closed process is a closed cone containing the origin.

For notational convenience, we introduce not only the set

$$\mathscr{P}(\mathbb{R}^n, \mathbb{R}^m) = \{F \mid F \text{ is a closed process from } \mathbb{R}^n \text{ to } \mathbb{R}^m\}$$

but also the following subsets

$$\mathscr{F}(\mathbb{R}^n, \mathbb{R}^m) = \{F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \mid F \text{ is a closed convex process}\},$$
$$\mathscr{F}_{\text{str}}(\mathbb{R}^n, \mathbb{R}^m) = \{F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \mid F \text{ is a strict closed convex process}\}.$$

Both subsets will play an important role in the discussion, especially the last one. Recall that a multivalued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called strict if its domain

$$D(F) = \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$$

is the whole space $\mathbb{R}^n$. That F is a closed convex process means that $\operatorname{gr} F$ is a closed convex cone. Convexity is an essential assumption in many of our statements.

The most common way of dealing with convergence issues in $\mathscr{P}(\mathbb{R}^n, \mathbb{R}^m)$ is by using the concept of Painlevé–Kuratowski convergence applied to the graphs: a sequence $\{F_\nu\}_{\nu\in\mathbb{N}}$ in $\mathscr{P}(\mathbb{R}^n, \mathbb{R}^m)$ is declared convergent if

$$\liminf_{\nu\to\infty}(\operatorname{gr} F_\nu) = \limsup_{\nu\to\infty}(\operatorname{gr} F_\nu), \tag{1.1}$$

where $\liminf_{\nu\to\infty} C_\nu$ and $\limsup_{\nu\to\infty} C_\nu$ denote, respectively, the lower and upper Painlevé–Kuratowski limits of a sequence $\{C_\nu\}_{\nu\in\mathbb{N}}$ of nonempty sets. For the sake of completeness, we recall the definition of these limits: if the C_ν are contained in some Euclidean space, say $\mathbb{R}^r$, then

$$\liminf_{\nu\to\infty} C_\nu = \left\{z \in \mathbb{R}^r \mid \lim_{\nu\to\infty} \operatorname{dist}[z, C_\nu] = 0\right\},$$
$$\limsup_{\nu\to\infty} C_\nu = \left\{z \in \mathbb{R}^r \mid \liminf_{\nu\to\infty} \operatorname{dist}[z, C_\nu] = 0\right\}.$$

In many practical situations, only one of these Painlevé–Kuratowski limits needs to be computed or scrutinized. In this paper, we pay special attention to the multivalued map

$$\liminf_{\nu\to\infty} F_\nu : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \tag{1.2}$$

defined by

$$\operatorname{gr}\Big(\liminf_{\nu\to\infty} F_\nu\Big) = \liminf_{\nu\to\infty}(\operatorname{gr} F_\nu).$$

In other words, we work mainly with the lower part of Painlevé–Kuratowski convergence and don't care too much about the upper part. With this approach, one doesn't burden oneself with the need of checking whether the equality (1.1) holds or not. The idea is that, for ν large enough, F_ν corresponds with an approximation or a perturbed version of (1.2). The latter map is used to model a reference or unperturbed multivalued system. Of course, one can see (1.2) also as a "limit" process.

An abstract and important question is whether $\liminf_{\nu\to\infty} F_\nu$ preserves the general properties of the approximations F_ν. Graph-closedness and positive homogeneity certainly yes, but what else? One may think, for instance, of graph-convexity. The answer is again yes! There are however plenty of properties that are not preserved. Some particularly bad news is this:

$$\text{if each } F_\nu \text{ is strict, it doesn't follow that } \liminf_{\nu\to\infty} F_\nu \text{ is strict as well.} \tag{1.3}$$

The observation (1.3) may seem quite irrelevant at first sight, but it is precisely the lack of strictness in the lower limit that is at the origin of many troubles, be they theoretical or computational.

In this paper, we deal, in fact, with a question that is somewhat related to the previous one:

$$\begin{cases} \text{knowing that } \liminf_{\nu\to\infty} F_\nu \text{ is strict,} \\ \text{what can be said about } F_\nu \text{ for } \nu \text{ large enough?} \end{cases}$$

We are not merely interested in guaranteeing the strictness of each F_ν, but we are also looking for the possibility of obtaining a uniform bound

$$\sup_{\nu\in\mathbb{N}} \|F_\nu\| < \infty \tag{1.4}$$

for the norms of the maps F_ν. This is a very subtle point that deserves a careful examination. There are several reasons why one should care about the uniform boundedness property (1.4). In Section 2, we illustrate this matter with the help of two illuminating examples.

2. WHY IS UNIFORM BOUNDEDNESS IMPORTANT?

Recall that the "norm" of a map $F \in \mathscr{P}(\mathbb{R}^n, \mathbb{R}^m)$ is understood as the nonnegative number

$$\|F\| = \sup_{\|x\| \le 1} \operatorname{dist}[0, F(x)]. \tag{2.1}$$

Observe that $\|F\| = \infty$ when F is not strict. Although (2.1) is not a norm in the usual sense of the word, at least it shares some of the properties of the operator (or spectral) norm $\|A\| = \sup_{\|x\| \le 1} \|Ax\|$ on the space of rectangular matrices. For general information on norms of positively homogeneous maps, see the book [19, Chapter 9]. Borwein's paper [5] is a good reference for norms of convex processes and their adjoints.

2.1. Pointwise Convergence versus Graphical Convergence

In general, the closed process $\liminf_{\nu \to \infty} F_\nu$ doesn't coincide with the map

$$x \in \mathbb{R}^n \rightrightarrows \liminf_{\nu \to \infty} [F_\nu(x)].$$

In the latter case, the priority is given to the images $\{F_\nu(x)\}_{\nu \in \mathbb{N}}$ and not to the graphs $\{\operatorname{gr} F_\nu\}_{\nu \in \mathbb{N}}$. Under convexity assumptions, however, a discrepancy between the maps $[\liminf_{\nu \to \infty} F_\nu](\cdot)$ and $\liminf_{\nu \to \infty} [F_\nu(\cdot)]$ cannot occur if the uniform boundedness condition (1.4) is in force. This result and the corresponding upper limit case is presented in the next proposition.

Proposition 2.1. *If* $\{F_\nu\}_{\nu \in \mathbb{N}}$ *is a sequence in* $\mathscr{F}_{\text{str}}(\mathbb{R}^n, \mathbb{R}^m)$ *satisfying the uniform boundedness condition* (1.4)*, then*

$$\Big[\liminf_{\nu \to \infty} F_\nu\Big](x) = \liminf_{\nu \to \infty} [F_\nu(x)], \tag{2.2}$$

$$\Big[\limsup_{\nu \to \infty} F_\nu\Big](x) = \limsup_{\nu \to \infty} [F_\nu(x)], \tag{2.3}$$

for any $x \in \mathbb{R}^n$.

Proof. This proposition is known and has been extended in several directions in the literature (cf. [3, 19]). We are presenting here a sketch of the proof only to illustrate the role played by the uniform boundedness condition (1.4). According to a theorem by Robinson [18], each element in $\mathscr{F}_{\text{str}}(\mathbb{R}^n, \mathbb{R}^m)$ has a finite norm. Not only that, but for each $F_\nu \in \mathscr{F}_{\text{str}}(\mathbb{R}^n, \mathbb{R}^m)$, one has

$$F_\nu(x') \subseteq F_\nu(x) + \|F_\nu\| \, \|x' - x\| \mathbb{B}_m \quad \forall x', x \in \mathbb{R}^n,$$

where $\mathbb{B}_m$ stands for the closed unit ball in $\mathbb{R}^m$. The interesting contribution of (1.4) is that the Lipschitz constant $\|F_\nu\|$ can be changed by something that doesn't depend on ν. Indeed, one can write

$$F_\nu(x') \subseteq F_\nu(x) + M\|x' - x\|\mathbb{B}_m \quad \forall x', x \in \mathbb{R}^n \tag{2.4}$$

with $M = \sup_{\nu\in\mathbb{N}} \|F_\nu\|$. Once the uniform Lipschitz estimate (2.4) has been established, the equalities (2.2) and (2.3) are obtained in a straightforward manner. □

2.2. Inner Stability of Reachable Sets

Let $R_T(F)$ denote the reachable set at time $T > 0$ associated with $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. By definition, a state $\xi \in \mathbb{R}^n$ belongs to $R_T(F)$ if and only if there is an absolutely continuous function $z : [0, T] \to \mathbb{R}^n$ solving the Cauchy problem

$$\begin{aligned} &\dot{z}(t) \in F(z(t)) \quad \text{a.e. on } [0, T] \\ &z(0) = 0 \end{aligned} \tag{2.5}$$

and such that $z(T) = \xi$. The set of all states that can be reached in finite time is of course

$$\text{Reach}(F) = \bigcup_{T>0} R_T(F).$$

As shown in the next proposition, the sets $R_T(F)$ and Reach(F) enjoy a certain stability property with respect to perturbations in the argument F.

Proposition 2.2. *Consider a sequence* $\{F_\nu\}_{\nu\in\mathbb{N}}$ *in* $\mathcal{F}_{\text{str}}(\mathbb{R}^n, \mathbb{R}^n)$ *satisfying the uniform boundedness condition* (1.4). *Then,*

$$R_T\Big(\liminf_{\nu\to\infty} F_\nu\Big) \subseteq \liminf_{\nu\to\infty} R_T(F_\nu) \quad \forall T > 0,$$

$$\text{Reach}\Big(\liminf_{\nu\to\infty} F_\nu\Big) \subseteq \liminf_{\nu\to\infty} \text{Reach}(F_\nu).$$

Proof. A more general result involving a delay factor in the differential inclusion (2.5) can be found in Lavilledieu and Seeger [12, Theorem 1]. The proof proposed by these authors is quite long and technical, so it is not worth recalling here the details. Suffice it to say that their proof is based on duality arguments and relies heavily on the uniform boundedness condition (1.4). □

3. UNIFORM BOUNDEDNESS AND STRICTNESS OF LOWER LIMITS

The main issue addressed in this paper is the analysis of the link existing between the strictness of the lower limit $\liminf_{\nu\to\infty} F_\nu$ and the boundedness of the sequence $\{\|F_\nu\|\}_{\nu\in\mathbb{N}}$. The following definition proves to be useful for a more concise presentation of our results.

Definition 3.1. A sequence $\{F_\nu\}_{\nu\in\mathbb{N}}$ in $\mathscr{P}(\mathbb{R}^n, \mathbb{R}^m)$ is called conditionally bounded if

$$\{\|F_\nu\|\}_{\nu\in\mathbb{N}} \text{ is bounded} \quad \text{or} \quad \liminf_{\nu\to\infty} F_\nu \text{ is not strict.}$$

Example 3.2. For each $\nu \in \mathbb{N}$, let $F_\nu : \mathbb{R}^n \rightrightarrows \mathbb{R}$ be given by $F_\nu(x) = \nu\|x\| + \mathbb{R}_+$. So, F_ν is strict and $\operatorname{gr} F_\nu = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \geq \nu\|x\|\}$ is a closed convex cone. Observe that $\{\|F_\nu\|\}_{\nu\in\mathbb{N}}$ is not bounded because $\|F_\nu\| = \nu$ goes to ∞. Anyway, $\{F_\nu\}_{\nu\in\mathbb{N}}$ is conditionally bounded because

$$x \in \mathbb{R}^n \rightrightarrows \left[\liminf_{\nu\to\infty} F_\nu\right](x) = \begin{cases} \mathbb{R}_+ & \text{if } x = 0, \\ \emptyset & \text{if } x \neq 0, \end{cases}$$

is not strict.

The term "conditional" has been chosen on purpose. The idea behind Definition 3.1 is that a conditionally bounded sequence $\{F_\nu\}_{\nu\in\mathbb{N}}$ is a sequence that is "bounded under the condition" that $\liminf_{\nu\to\infty} F_\nu$ is strict. This is not just a matter of playing with words. In many cases, it is easy to check whether $\liminf_{\nu\to\infty} F_\nu$ is strict or not. Once this point has been clarified, one checks then the uniform boundedness property (1.4).

It is important and useful to build up a battery of examples of conditionally bounded sequences. We start with an observation that is not completely obvious after all: sequences that are not conditionally bounded do exist!

Example 3.3. Consider the closed processes $F_\nu : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ defined by

$$F_\nu(x_1, x_2) = \begin{cases} 0 & \text{if } \nu|x_2| \geq x_1, \\ \nu\big(1 - \nu x_1^{-1}|x_2|\big)\big(x_1 + |x_2|\big) & \text{if } \nu|x_2| < x_1. \end{cases}$$

Evidently, each F_ν is strict. As the F_ν are single-valued, they may be considered as ordinary functions. These functions are continuous and positively homogeneous. As a consequence, the sets $\operatorname{gr} F_\nu$ are closed cones, so the F_ν are closed processes indeed. Because $\operatorname{dist}[0, F_\nu(1, 0)] = \nu$, it follows that $\|F_\nu\| \to \infty$. On the other hand, $F = \liminf_{\nu\to\infty} F_\nu$ is strict. In fact, one

FIGURE 1 Illustration of $\operatorname{gr} F_\nu$ for $\nu = 5$ (left) and $\nu = 10$ (center) and of $\operatorname{gr} F$ (right).

can easily check that $0 \in F(x)$ for all $x \in \mathbb{R}^2$. In conclusion, the sequence $\{F_\nu\}_{\nu\in\mathbb{N}}$ is not conditionally bounded (cf. Fig. 1).

Already at this point, the reader might guess that the lack of convexity in $\operatorname{gr} F_\nu$ is at the origin of this trouble with Example 3.3. As we shall see later in Section 5.1, convexity helps indeed in securing conditional boundedness.

We state below two minor results on the preservation of conditional boundedness under perturbations on the data.

Proposition 3.4. *Let $\{F_\nu\}_{\nu\in\mathbb{N}}$ and $\{G_\nu\}_{\nu\in\mathbb{N}}$ be two sequences in $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^m)$ related by*

$$F_\nu(x) = Q_\nu[G_\nu(x)] \quad \forall x \in \mathbb{R}^n,$$

where $\{Q_\nu\}_{\nu\in\mathbb{N}}$ is a sequence of $m \times m$ matrices converging to a nonsingular matrix. Then, $\{F_\nu\}_{\nu\in\mathbb{N}}$ is conditionally bounded if and only if $\{G_\nu\}_{\nu\in\mathbb{N}}$ is conditionally bounded.

Proof. Boundedness of $\{\|F_\nu\|\}_{\nu\in\mathbb{N}}$ is equivalent to boundedness of $\{\|G_\nu\|\}_{\nu\in\mathbb{N}}$. To see this, it is enough to write

$$\|F_\nu\| \leq \|Q_\nu\| \, \|G_\nu\| \quad \forall \nu \in \mathbb{N},$$
$$\|G_\nu\| \leq \|Q_\nu^{-1}\| \, \|F_\nu\| \quad \forall \nu \geq \nu_0,$$

where ν_0 is the smallest integer such that Q_ν is invertible for all $\nu \geq \nu_0$. On the other hand, one can show that

$$D\Big(\liminf_{\nu\to\infty} F_\nu\Big) = D\Big(\liminf_{\nu\to\infty} G_\nu\Big). \tag{3.1}$$

The proof of (3.1) is more or less straightforward, so we omit the details. Notice that (3.1) implies in particular that $\liminf_{\nu\to\infty} F_\nu$ is strict if and only if $\liminf_{\nu\to\infty} G_\nu$ is strict. The proof of the proposition is then complete. □

Proposition 3.5. *Let $\{F_\nu\}_{\nu\in\mathbb{N}}$ and $\{G_\nu\}_{\nu\in\mathbb{N}}$ be two sequences in $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^m)$ related by*

$$F_\nu(x) = G_\nu(x) + P_\nu x \quad \forall x \in \mathbb{R}^n,$$

where $\{P_\nu\}_{\nu\in\mathbb{N}}$ is a converging sequence of matrices of size $m \times n$. Then, $\{F_\nu\}_{\nu\in\mathbb{N}}$ is conditionally bounded if and only if $\{G_\nu\}_{\nu\in\mathbb{N}}$ is conditionally bounded.

Proof. It is essentially the same proof as in Proposition 3.4. □

4. BUNDLES OF LINEAR OPERATORS

A (compact) bundle of linear operators is an important example of closed process. The formal definition of a bundle is as follows:

Definition 4.1. Let Ξ be a nonempty compact set in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$. The bundle associated with Ξ is the multivalued map $F^\Xi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by

$$F^\Xi(x) = \{Ax \mid A \in \Xi\} \quad \forall x \in \mathbb{R}^n.$$

It goes without saying that $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, can be identified with the space of real matrices of size $m \times n$. We are asking Ξ to be compact only for the sake of simplicity in the presentation. The compactness of Ξ ensures in particular that $\operatorname{gr} F^\Xi$ is a closed set. Positive homogeneity of F^Ξ is obvious and requires no assumption on Ξ.

Bundles admit a large variety of applications and have been extensively studied in the literature (see [2] and references therein). In connection with this class of maps, we would like to explore the following question:

$$\begin{cases} \text{what kind of hypothesis on the sets } \{\Xi_\nu\}_{\nu\in\mathbb{N}} \subset \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m) \\ \text{would ensure the conditional boundedness of } \{F^{\Xi_\nu}\}_{\nu\in\mathbb{N}}? \end{cases}$$

Before answering this question, we have to set straight a couple of things. First of all, the norm of any bundle is finite. In fact, directly from the definition (2.1) of the number $\|F^\Xi\|$, one sees that

$$\|F^\Xi\| = \sup_{\|x\|\le 1} \inf_{A\in\Xi} \|Ax\| \le \inf_{A\in\Xi} \sup_{\|x\|\le 1} \|Ax\| = \inf_{A\in\Xi} \|A\| = \operatorname{dist}[0, \Xi]. \tag{4.1}$$

Under special circumstances, it is possible to exchange the order of the supremum and the infimum in (4.1), but this is something we don't need to address here. The important point is that

$$\|F^\Xi\| \le \operatorname{dist}[0, \Xi] < \infty \quad \text{for any } \Xi.$$

Regarding the asymptotic behavior of the sets $\{\Xi_\nu\}_{\nu\in\mathbb{N}}$, two mutually exclusive cases are to be considered:

$$\sup_{\nu\in\mathbb{N}} \operatorname{dist}[0, \Xi_\nu] < \infty, \tag{4.2}$$

$$\limsup_{\nu\to\infty} \operatorname{dist}[0, \Xi_\nu] = \infty. \tag{4.3}$$

In the first case, the sequence $\{\|F^{\Xi_\nu}\|\}_{\nu\in\mathbb{N}}$ is necessarily bounded, and there is nothing more to discuss. The second case is of course more interesting.

The next definition is a bit technical and needs some clarifications. We use the notation A^T to indicate the transpose of a given $A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$. The symbol e_i stands for the ith canonical vector in a certain Euclidean space of appropriate dimension. Consistently with standard matrix algebra, $A^T e_i$ corresponds with the ith row of A.

Definition 4.2. Consider a sequence $\vec{\Xi} = \{\Xi_\nu\}_{\nu\in\mathbb{N}}$ of nonempty sets in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$, be they compact or not. A unit vector $w \in \mathbb{R}^n$ is called a recession direction of $\vec{\Xi}$ if there is sequence $\{A_\nu\}_{\nu\in\mathbb{N}}$ such that

$$A_\nu \in \Xi_\nu \quad \forall \nu \in \mathbb{N},$$

$$\lim_{\nu\to\infty} \left\|A^T_{\varphi(\nu)} e_i\right\| = \infty,$$

$$\lim_{\nu\to\infty} \frac{A^T_{\varphi(\nu)} e_i}{\left\|A^T_{\varphi(\nu)} e_i\right\|} = w,$$

for some increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ and some row $i \in \{1, \ldots, m\}$. The set of all recession directions of $\vec{\Xi}$ is denoted by $\operatorname{rec}[\vec{\Xi}]$.

Definition 4.2 is specially meaningful when the condition (4.3) is in force. Such condition ensures in fact the nonvacuity of $\operatorname{rec}[\vec{\Xi}]$.

Example 4.3. For each $\nu \in \mathbb{N}$, let the bundle $F_\nu : \mathbb{R}^3 \rightrightarrows \mathbb{R}^2$ be given by

$$F_\nu(x) = \left\{ \underbrace{\begin{bmatrix} 1 & 3\nu & 2 \\ -1 & 4 & 0 \end{bmatrix}}_{A_\nu} x, \underbrace{\begin{bmatrix} 2 & -3 & 1 \\ \nu & 1 & \nu^2 \end{bmatrix}}_{B_\nu} x, \underbrace{\begin{bmatrix} 1 & 2 & \frac{2}{\nu+1} \\ -4 & 1 & 2 \end{bmatrix}}_{C_\nu} x \right\} \quad \forall x \in \mathbb{R}^3.$$

The second row of A_ν remains bounded as $\nu \to \infty$, so this row doesn't produce recession directions. The same remark applies to the first row of

B_ν and to both rows of C_ν. By contrast, the first row of A_ν produces the recession direction

$$(0,1,0) = \lim_{\nu\to\infty} \frac{(1,3\nu,2)}{\sqrt{1+9\nu^2+4}},$$

whereas the second row of B_ν produces

$$(0,0,1) = \lim_{\nu\to\infty} \frac{(\nu,1,\nu^2)}{\sqrt{\nu^2+1+\nu^4}}.$$

In this example, one has $\mathrm{rec}[\vec{\Xi}] = \{(0,1,0),(0,0,1)\}$.

In what follows, we use the symbol $w^\perp$ for indicating the hyperplane that is orthogonal to w.

Lemma 4.4. *Let $\vec{\Xi} = \{\Xi_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of nonempty compact sets in $\mathscr{L}(\mathbb{R}^n,\mathbb{R}^m)$. If the condition* (4.3) *is in force, then*

$$D\Big(\liminf_{\nu\to\infty} F^{\Xi_\nu}\Big) \subset \bigcup_{w\in\mathrm{rec}[\vec{\Xi}]} w^\perp. \tag{4.4}$$

Proof. Let x be in the left-hand side of (4.4). Pick any $y \in [\liminf_{\nu\to\infty} F^{\Xi_\nu}](x)$. By definition of the lower limit, there are sequences $\{x_\nu\}_{\nu\in\mathbb{N}} \to x$, $\{y_\nu\}_{\nu\in\mathbb{N}} \to y$, and $\{A_\nu\}_{\nu\in\mathbb{N}}$ such that

$$\begin{aligned} &A_\nu \in \Xi_\nu \\ &y_\nu = A_\nu x_\nu \end{aligned} \tag{4.5}$$

for all $\nu \in \mathbb{N}$. By combining $\mathrm{dist}[0,\Xi_\nu] \le \|A_\nu\|$ and assumption (4.3), one sees that $\{A_\nu\}_{\nu\in\mathbb{N}}$ is unbounded. Taking a subsequence if necessary, one may assume that $\|A_\nu^T e_i\| \to \infty$ for some row $i \in \{1,\ldots,m\}$. Taking yet another subsequence allows us to write

$$\frac{A_\nu^T e_i}{\|A_\nu^T e_i\|} \to w$$

for some unit vector $w \in \mathbb{R}^n$. By construction, w belongs to $\mathrm{rec}[\vec{\Xi}]$. Now, a division by $\|A_\nu^T e_i\|$ in the ith equation of (4.5) yields

$$\frac{\langle e_i, y_\nu\rangle}{\|A_\nu^T e_i\|} = \left\langle \frac{A_\nu^T e_i}{\|A_\nu^T e_i\|}, x_\nu \right\rangle.$$

By passing to the limit in the above line, one gets $0 = \langle w, x\rangle$, proving in this way that x belongs to the right-hand side of (4.4). □

Theorem 4.5. *Let $\vec{\Xi} = \{\Xi_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of nonempty compact sets in $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that*

$$\bigcup_{w\in\mathrm{rec}[\vec{\Xi}]} w^\perp \neq \mathbb{R}^n. \tag{4.6}$$

Then, $\{F^{\Xi_\nu}\}_{\nu\in\mathbb{N}}$ is conditionally bounded.

Proof. If (4.2) holds, then $\{\|F^{\Xi_\nu}\|\}_{\nu\in\mathbb{N}}$ is bounded and we are done. Otherwise, (4.3) is in force, and Lemma 4.4 yields the announced result. Indeed, (4.6) prevents the map $\liminf_{\nu\to\infty} F^{\Xi_\nu}$ from being strict. □

Remark 4.6. Assumption (4.6) holds, for instance, if $\mathrm{rec}[\vec{\Xi}]$ is finite or countable. Indeed, the whole space $\mathbb{R}^n$ cannot be recovered by taking a union of finitely or countably many hyperplanes. Example 4.3 falls of course into this category.

5. CONVEX PROCESSES

Positive homogeneity alone is sometimes not rich enough as a working hypothesis. Convexity adds substantial structure to the data and allows the use of separation arguments among other tools.

The next lemma is probably known. Its proof is given only for the sake of completeness.

Lemma 5.1. *Consider a sequence of closed convex processes $F_\nu : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ whose lower limit $F = \liminf_{\nu\to\infty} F_\nu$ is strict. Then, there exists an integer $\nu_0 \in \mathbb{N}$ such that the F_ν are strict for all $\nu \geq \nu_0$.*

Proof. From the very definition of the Painlevé–Kuratowski limits, one can easily check that

$$D(F) \subset \liminf_{\nu\to\infty} D(F_\nu) \subset \liminf_{\nu\to\infty} \mathrm{cl}[D(F_\nu)] \subset \limsup_{\nu\to\infty} \mathrm{cl}[D(F_\nu)]$$

with "cl" standing for topological closure. Because F is assumed to be strict, one ends up with

$$\liminf_{\nu\to\infty}\ \mathrm{cl}[D(F_\nu)] = \limsup_{\nu\to\infty}\ \mathrm{cl}[D(F_\nu)] = \mathbb{R}^n.$$

In short, $\mathrm{cl}[D(F_\nu)]$ are closed convex cones converging to $\mathbb{R}^n$. A simple separation argument implies that $\mathrm{cl}[D(F_\nu)] = \mathbb{R}^n$ for all ν large enough. This proves the assertion because the closure operation can be dropped in the above line. □

Remark 5.2. Lemma 5.1 has more to do with domains than with graphs. Instead of asking F_v to be a closed convex process, it suffices asking F_v to be a closed process with convex domain.

5.1. Conditional Boundedness as Consequence of Convexity

Under the assumptions of Lemma 5.1, one gets immediately

$$\|F_v\| < \infty \quad \text{for all } v \text{ large enough,}$$

but, in fact, it is possible to derive a much stronger conclusion. The next theorem is one of the main results of this paper.

Theorem 5.3. *Consider a sequence of closed convex processes* $F_v : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *whose lower limit* $F = \liminf_{v\to\infty} F_v$ *is strict. Then,*

(a) $\limsup_{v\to\infty} \|F_v\| \leq \|F\|$,
(b) *there exists an integer* $v_0 \in \mathbb{N}$ *such that the tail* $\{\|F_v\|\}_{v\geq v_0}$ *is bounded.*

Proof. The part (b) is of course a consequence of (a). For the proof of (a) suppose, on the contrary, that there are a subsequence $\{\|F_v\|\}_{v\in N_1}$ and a real $\beta > 0$ such that

$$\|F_v\| \geq \|F\| + \beta \quad \forall v \in N_1. \tag{5.1}$$

In view of Lemma 5.1, we may assume that F_v is strict and

$$\|F_v\| = \sup_{\|x\|=1} \operatorname{dist}[0, F_v(x)] < \infty \tag{5.2}$$

for all $v \in N_1$. Notice that the condition (5.2) alone doesn't prevent $\{\|F_v\|\}_{v\in N_1}$ from diverging to ∞. For each $v \in N_1$, pick a pair $(x_v, y_v) \in \operatorname{gr} F_v$ such that

$$\|x_v\| = 1, \quad \|y_v\| = \operatorname{dist}[0, F_v(x_v)] \geq \|F_v\| - (1/v). \tag{5.3}$$

By compactness of the unit sphere, there is another subsequence $\{x_v\}_{v\in N_2}$ that converges to some unit vector, say $\bar{x}$. Because F is strict by assumption, we may choose some $\bar{y} \in F(\bar{x})$. In fact, we take $\bar{y}$ as the least-norm element of $F(\bar{x})$ in order to obtain

$$\|\bar{y}\| \leq \|F\|. \tag{5.4}$$

By definition of the lower limit F, one can write

$$(\bar{x}, \bar{y}) = \lim_{v\to\infty} (\bar{x}_v, \bar{y}_v)$$

for some sequence $\{(\bar{x}_\nu, \bar{y}_\nu)\}_{\nu\in\mathbb{N}}$ such that $(\bar{x}_\nu, \bar{y}_\nu) \in \operatorname{gr} F_\nu$ for all ν large enough. Of course, the subsequence $\{(\bar{x}_\nu, \bar{y}_\nu)\}_{\nu\in N_2}$ converges to the same limit $(\bar{x}, \bar{y})$. Dropping the first indices in N_2 if necessary, we may suppose that

$$\|\bar{y}_\nu - \bar{y}\| \leq \beta/2 \quad \forall \nu \in N_2. \tag{5.5}$$

The combination of (5.4) and (5.5) produces

$$\|\bar{y}_\nu\| \leq \|F\| + (\beta/2) \quad \forall \nu \in N_2. \tag{5.6}$$

Next, for each $\nu \in N_2$, define $s_\nu = 2x_\nu - \bar{x}_\nu$. Observe that $\{s_\nu\}_{\nu\in N_2}$ converges to $\bar{x}$. We claim that

$$\operatorname{dist}[0, F_\nu(s_\nu)] \geq 2\|y_\nu\| - \|\bar{y}_\nu\| \tag{5.7}$$

for all $\nu \in N_2$. Pick an arbitrary $w \in F_\nu(s_\nu)$ (note that there is at least one due to the F_ν being strict). By convexity of $\operatorname{gr} F_\nu$, one gets that

$$\frac{1}{2}(\bar{x}_\nu + s_\nu, \bar{y}_\nu + w) = \left(x_\nu, \frac{\bar{y}_\nu + w}{2}\right) \in \operatorname{gr} F_\nu.$$

Consequently,

$$\|y_\nu\| = \operatorname{dist}[0, F_\nu(x_\nu)] \leq \frac{1}{2}\|\bar{y}_\nu + w\| \leq \frac{1}{2}(\|\bar{y}_\nu\| + \|w\|),$$

and therefore $\|w\| \geq 2\|y_\nu\| - \|\bar{y}_\nu\|$. The inequality (5.7) is obtained by taking the infimum with respect to $w \in F_\nu(s_\nu)$. We now invoke the positive homogeneity of F_ν in order to write

$$\|F_\nu\| \geq \operatorname{dist}\left[0, F_\nu\left(\frac{s_\nu}{\|s_\nu\|}\right)\right] \geq \frac{2\|y_\nu\| - \|\bar{y}_\nu\|}{\|s_\nu\|}.$$

In view of (5.3), one gets

$$\|F_\nu\| \geq \frac{2[\|F_\nu\| - (1/\nu)] - \|\bar{y}_\nu\|}{\|s_\nu\|}. \tag{5.8}$$

The first conclusion that can be drawn from (5.8) is that $\{\|F_\nu\|\}_{\nu\in N_2}$ is bounded. This can be easily seen by writing (5.8) in the equivalent form

$$\|F_\nu\| \leq \frac{(2/\nu) + \|\bar{y}_\nu\|}{2 - \|s_\nu\|}$$

because the right-hand side converges to $\|\bar{y}\|$. The combination of (5.1) and (5.6) yields

$$-\|\bar{y}_\nu\| \geq (\beta/2) - \|F_\nu\|.$$

Plugging this information in (5.8), one obtains

$$\|F_\nu\| \geq \frac{2[\|F_\nu\| - (1/\nu)] + (\beta/2) - \|F_\nu\|}{\|s_\nu\|}.$$

After a short rearrangement, one gets

$$(\|s_\nu\| - 1)\|F_\nu\| \geq (\beta/2) - (2/\nu).$$

By taking $\nu \in N_2$ sufficiently large, one arrives at a contradiction because $\|s_\nu\| \to 1$ and $\{F_\nu\}_{\nu \in N_2}$ is bounded. The proof of the theorem is thus complete. □

Corollary 5.4. *Any sequence of strict closed convex processes $F_\nu : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is conditionally bounded.*

Proof. If $\liminf_{\nu\to\infty} F_\nu$ is not strict, then there is nothing to prove. Otherwise, there exists a positive integer ν_0 such that the tail $\{\|F_\nu\|\}_{\nu\geq\nu_0}$ is bounded. The first terms $\{\|F_\nu\|\}_{\nu\leq\nu_0-1}$ form a finite collection of real numbers because all the F_ν are assumed to be strict. Hence, the whole sequence $\{\|F_\nu\|\}_{\nu\in\mathbb{N}}$ is bounded. □

5.2. A Continuity Result for the Operator Norm on $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$

Under the assumptions of Theorem 5.3, it may well happen that

$$\|F\| > \limsup_{\nu\to\infty} \|F_\nu\|, \tag{5.9}$$

with the right-hand side in (5.9) being even a usual limit. The next example illustrates this point.

Example 5.5. For each $\nu \in \mathbb{N}$, consider the closed convex process $F_\nu : \mathbb{R} \rightrightarrows \mathbb{R}^2$ given by

$$\operatorname{gr} F_\nu = \{(x, y_1, y_2) \mid y_1 \geq 0, (-1)^\nu y_2 \geq 0, 2x + y_1 + 2(-1)^\nu y_2 \geq 0\}.$$

A matter of computation shows that $F = \liminf_{\nu\to\infty} F_\nu$ has

$$\operatorname{gr} F = \{(x, y_1, y_2) \mid y_1 \geq 0, y_2 = 0, 2x + y_1 \geq 0\}$$

as graph. Hence, F is strict and $\|F\| = 2$. On the other hand, $\|F_\nu\| = 2/\sqrt{5}$ for all $\nu \in \mathbb{N}$, and therefore $\limsup_{\nu\to\infty} \|F_\nu\| = \liminf_{\nu\to\infty} \|F_\nu\| = 2/\sqrt{5}$.

Example 5.5 is not too surprising after all if one takes into account that $\{F_\nu\}_{\nu\in\mathbb{N}}$ doesn't converge in the Painlevé–Kuratowski sense.

A multivalued map can always be identified with its graph. Thus, $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ is indistinguishable from the class of all nonempty closed convex cones in $\mathbb{R}^{n+m}$. This observation leads us to measure the distance between two closed convex processes $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ by means of the expression

$$\delta(G, F) = \text{haus}[(\text{gr}\, G) \cap \mathbb{B}_{n+m}, (\text{gr}\, F) \cap \mathbb{B}_{n+m}]$$

with $\text{haus}(C, D)$ denoting the classical Pompeiu–Hausdorff distance between two nonempty closed bounded sets C, D. By an obvious reason, one refers to δ as the truncated Pompeiu–Hausdorff metric.

Painlevé–Kuratowski convergence in $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ is the same thing as convergence with respect to the metric δ (cf. [19, Chapter 4]). The following continuity result is obtained as consequence of Theorem 5.3.

Corollary 5.6. *Let $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ be equipped with the metric δ (or with any other metric that induces Painlevé–Kuratowski convergence). Then, the operator norm*

$$\|\cdot\| : \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R} \cup \{\infty\}$$

is finite-valued and continuous at each F that is strict.

Proof. Let $F \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ be strict. Take any sequence $\{F_\nu\}_{\nu\in\mathbb{N}}$ in $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\delta(F_\nu, F) \to 0 \quad \text{as } \nu \to \infty. \tag{5.10}$$

The inequality $\limsup_{\nu\to\infty} \|F_\nu\| \le \|F\|$ has been already established in Theorem 5.3(a), so it remains to prove that

$$\|F\| \le \liminf_{\nu\to\infty} \|F_\nu\|. \tag{5.11}$$

In view of Theorem 5.3(b), there is a positive integer ν_0 such that $\{\|F_\nu\|\}_{\nu\ge\nu_0}$ is bounded. In particular, each F_ν is strict. Notice that $\gamma = \liminf_{\nu\to\infty} \|F_\nu\|$ is a finite number and, on the other hand, it is possible to write

$$F_\nu(x') \subseteq F_\nu(x) + \|F_\nu\| \|x' - x\| \mathbb{B}_m$$

for all $x', x \in \mathbb{R}^n$ and $v \geq v_0$. By taking lower Painlevé–Kuratowski limits on each side of the above inclusion, one gets

$$\liminf_{v\to\infty} [F_v(x')] \subseteq \liminf_{v\to\infty} [F_v(x) + \|F_v\| \, \|x' - x\| \mathbb{B}_m].$$

By applying Proposition 2.1 and the convergence assumption (5.10), one arrives finally at

$$F(x') \subseteq F(x) + \gamma \, \|x' - x\| \mathbb{B}_m \quad \forall x', x \in \mathbb{R}^n.$$

The particular choice $x' = 0$ yields

$$0 \in F(x) + \gamma \|x\| \mathbb{B}_m \quad \forall x \in \mathbb{R}^n,$$

which, in turn, proves (5.11). □

5.3. The Case of Linear Relations

A map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called a linear relation if its graph is a linear subspace. Norms of linear relations in infinite dimensional spaces have been studied with great care by Lee and Nashed [14]. We propose a somewhat different technique for representing a linear relation and for computing its norm. We restrict the attention to a finite dimensional setting.

Lemma 5.7. *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a strict multivalued map whose graph is a linear subspace, say of dimension $r < n + m$. Then, $\operatorname{gr} F$ can be represented in the form*

$$\operatorname{gr} F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid Ax + By = 0\}, \tag{5.12}$$

where A is a matrix of size $(n + m - r) \times n$ and B is a surjective matrix of size $(n + m - r) \times m$. Furthermore, $\|F\| = \|B^T (BB^T)^{-1} A\|$.

Proof. Because $\operatorname{gr} F$ is a linear subspace of dimension r, there exist matrices A and B of the indicated sizes such that the joint matrix $[A \mid B]$ is surjective and

$$\operatorname{gr} F = \ker[A \mid B].$$

We claim that B is surjective. *Ab absurdo,* suppose that there exists a vector $\bar{z} \in \mathbb{R}^{n+m-r}$ such that $Bw \neq \bar{z}$ for all $w \in \mathbb{R}^m$. The matrix $[A \mid B]$ being surjective, there is some $(\bar{x}, \bar{y}) \in \mathbb{R}^{n+m}$ such that $A\bar{x} + B\bar{y} = \bar{z}$. It follows that $A\bar{x} + B\bar{y} \neq Bw$ for all $w \in \mathbb{R}^m$, whence $A\bar{x} + By \neq 0$ for all $y \in \mathbb{R}^m$. Given the description of $\operatorname{gr} F$ by means of A and B, this amounts to saying that $(\bar{x}, y) \notin \operatorname{gr} F$ for all $y \in \mathbb{R}^m$. In other words, $F(\bar{x}) = \emptyset$, which contradicts the assumption of F being strict. Concerning the second part of the lemma, note that for any $x \in \mathbb{R}^n$, the image

$$F(x) = \{y \in \mathbb{R}^m | By = -Ax\}$$

represents an affine subspace of $\mathbb{R}^m$, the associated linear subspace being the kernel of B. The norm-minimal element of $F(x)$ is the projection

$$\operatorname{Proj}[0, F(x)] = -B^T(BB^T)^{-1}Ax.$$

Hence,

$$\begin{aligned} \|F\| &= \sup_{\|x\|\le 1} \operatorname{dist}[0, F(x)] = \sup_{\|x\|\le 1} \| - B^T(BB^T)^{-1}Ax\| \\ &= \|B^T(BB^T)^{-1}A\|. \end{aligned}$$

This completes the proof of the lemma. □

Theorem 5.3 can be specialized to the case of linear relations. Consider a sequence of closed convex processes $F_\nu : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ converging in the Painlevé–Kuratowski sense to a map F that is strict. Suppose, in addition, that the graph of F is a linear subspace of dimension $r < n + m$. In such a case, one can draw the following conclusions:

- For all $\nu \ge \nu_0$, with $\nu_0 \in \mathbb{N}$ large enough, the map F_ν is a strict linear relation. Furthermore, $\dim[\operatorname{gr} F_\nu] = r$.
- The graph of F_ν admits the representation (5.12) with suitable matrices A_ν and B_ν as in Lemma 5.7.
- The sequence $\{\|B_\nu^T(B_\nu B_\nu^T)^{-1}A_\nu\|\}_{\nu\ge\nu_0}$ is bounded.

The first conclusion is a consequence of Lemma 5.1 and [10, Proposition 6.3]. The second and third conclusions are obtained by combining Lemma 5.7 and Theorem 5.3.

5.4. A Counterexample in Infinite Dimension

Finite dimensionality is an essential assumption in Theorem 5.3. Consider, for instance, the Hilbert space

$$l^2(\mathbb{R}) = \left\{x : \mathbb{N} \to \mathbb{R} \,\middle|\, \sum_{i\in\mathbb{N}} [x(i)]^2 < \infty\right\},$$

and the closed convex processes $F_\nu : l^2(\mathbb{R}) \rightrightarrows \mathbb{R}$ given by

$$F_\nu(x) = \begin{cases} \mathbb{R}_+ & \text{if } x(\nu) \le 0 \\ \nu x(\nu) + \mathbb{R}_+ & \text{if } x(\nu) > 0. \end{cases}$$

The sequence $\{F_\nu\}_{\nu\in\mathbb{N}}$ converges in the Painlevé–Kuratowski sense to the closed convex process $x \in l^2(\mathbb{R}) \rightrightarrows F(x) = \mathbb{R}_+$. This can be proven by verifying the inclusions

$$\Big[\limsup_{\nu\to\infty} F_\nu\Big](\bar{x}) \subseteq F(\bar{x}) \subseteq \Big[\liminf_{\nu\to\infty} F_\nu\Big](\bar{x})$$

at some reference point $\bar{x} \in l^2(\mathbb{R})$. Let $\bar{y} \in (\limsup_{\nu\to\infty} F_\nu)(\bar{x})$. Then, there is some $N_1 \subseteq \mathbb{N}$ such that $\{(x_\nu, y_\nu)\}_{\nu\in N_1} \to (\bar{x}, \bar{y})$ and $y_\nu \in F_\nu(x_\nu)$ for all $\nu \in N_1$. By definition of F_ν, one has $y_\nu \ge 0$. Consequently, $\bar{y} \in F(\bar{x})$, as was to be shown. For the second inclusion, let $\bar{y} \in F(\bar{x})$ be given. Define $\{x_\nu\}_{\nu\in\mathbb{N}}$ and $\{y_\nu\}_{\nu\in\mathbb{N}}$ by

$$x_\nu(i) = \begin{cases} \bar{x}(i) & \text{if } i < \nu \\ 0 & \text{if } i \ge \nu, \end{cases}$$

and $y_\nu = \bar{y}$, respectively. Then, $F_\nu(x_\nu) = \mathbb{R}_+$, because $x_\nu(\nu) = 0$. Hence, $y_\nu \in F_\nu(x_\nu)$. Moreover,

$$\|x_\nu - \bar{x}\|^2 = \sum_{i=\nu}^{\infty} [\bar{x}(i)]^2 \to 0 \quad \text{as } \nu \to \infty.$$

This completes the proof of the second inclusion.

Despite the fact that F is strict, the sequence $\{\|F_\nu\|\}_{\nu\in\mathbb{N}}$ is not bounded. To see this, consider $\{x_\nu\}_{\nu\in\mathbb{N}}$ given by

$$x_\nu(i) = \begin{cases} 1 & \text{if } i = \nu \\ 0 & \text{if } i \ne \nu. \end{cases}$$

Clearly, $\|x_\nu\| = 1$ for all $\nu \in \mathbb{N}$. Hence,

$$\|F_\nu\| \ge \operatorname{dist}[0, F_\nu(x_\nu)] = \operatorname{dist}[0, \nu + \mathbb{R}_+] = \nu$$

goes to infinity.

6. AN APPLICATION TO CONTROL THEORY

This section deals with the controllability of a differential inclusion

$$\dot{z}(t) \in F(z(t)) \quad \text{a.e. on } [0, T] \tag{6.1}$$

whose right-hand side is a closed convex process $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Differential inclusions governed by convex processes arise in many areas of applied mathematics. Readers wishing to know more about this particular topic are offered a short introduction to relevant literature at the end of the paper; see the Appendix. Recall that $F \in \mathscr{F}(\mathbb{R}^n, \mathbb{R}^n)$ is said to be controllable if

$$\begin{cases} \text{for all } \xi \in \mathbb{R}^n \text{ there is an absolutely continuous function} \\ z : [0, T] \to \mathbb{R}^n \text{ satisfying the differential inclusion (6.1),} \\ \text{and the end-point conditions } z(0) = 0 \text{ and } z(T) = \xi. \end{cases}$$

We would like to know what happens with the controllability of (6.1) if the map F is slightly perturbed. Robustness of controllability for a system like (6.1) is a topic that has been studied by Naselli-Ricceri [16], Tuan [22], Lavilledieu and Seeger [13], and Henrion et al. [8], among other authors.

Theorem 6.3 below shows that controllability is a robust concept. The proof of this result relies on the characterization of controllability stated in Lemma 6.2. First we write:

Definition 6.1. Let $Q \subseteq \mathbb{R}^n$ be a closed convex cone and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a closed convex process. One says that Q is *invariant* by F if $F(Q) \subseteq Q$.

Lemma 6.2 (cf. [4]). *Let $F \in \mathscr{F}(\mathbb{R}^n, \mathbb{R}^n)$. Then, F is controllable if and only if F is strict and the space $\mathbb{R}^n$ is the only closed convex cone that is invariant by F.*

Without further ado, we state:

Theorem 6.3. *Consider a sequence $\{F_v\}_{v\in\mathbb{N}}$ in $\mathscr{F}(\mathbb{R}^n, \mathbb{R}^n)$ such that $\liminf_{v\to\infty} F_v$ is controllable. Then, for all v large enough, F_v is controllable as well.*

Proof. Let $F = \liminf_{v\to\infty} F_v$. Controllability of F implies strictness of F. By Lemma 5.1 and Theorem 5.3, there are a positive integer v_0 and a constant M_0 such that

$$F_v \text{ is strict} \quad \text{and} \quad \|F_v\| \le M_0$$

for all $v \ge v_0$. From now on we work with the tail $\{F_v\}_{v\ge v_0}$. By contraposition, suppose that this tail admits a subsequence $\{F_v\}_{v\in N_1}$ of uncontrollable processes. We must arrive at a contradiction. By Lemma 6.2, for each $v \in N_1$, there is a closed convex cone $Q_v \neq \mathbb{R}^n$ such that

$$F_v(Q_v) \subseteq Q_v. \tag{6.2}$$

Taking another subsequence if necessary, one may assume that $\{Q_\nu\}_{\nu\in N_2\subseteq N_1}$ converges to some closed convex cone $Q \subseteq \mathbb{R}^n$. We are invoking here the fact that every sequence in

$$\mathscr{K}(\mathbb{R}^n) = \{K \subseteq \mathbb{R}^n \mid K \text{ is a closed convex cone}\}$$

admits a subsequence that converges in the Painlevé–Kuratowski sense toward an element of $\mathscr{K}(\mathbb{R}^n)$ (cf. [10, Proposition 2.1]). Because each Q_ν is different from $\mathbb{R}^n$, so is the cone Q. For completing the proof, it is enough to show that Q is invariant by F because this would contradict the controllability of F. Take $x \in Q$ and $y \in F(x)$. There are sequences

$$\{(x_\nu, y_\nu)\}_{\nu\in N_2} \to (x, y), \quad \{\hat{x}_\nu\}_{\nu\in N_2} \to x$$

such that $(x_\nu, y_\nu) \in \operatorname{gr} F_\nu$ and $\hat{x}_\nu \in Q_\nu$. By strictness of F_ν, one has the Lipschitz estimate

$$F_\nu(x_\nu) \subseteq F_\nu(\hat{x}_\nu) + \|F_\nu\| \, \|x_\nu - \hat{x}_\nu\| \mathbb{B}_n.$$

Because the norms of the F_ν are majorized by M_0, one gets

$$F_\nu(x_\nu) \subseteq F_\nu(\hat{x}_\nu) + M_0 \|x_\nu - \hat{x}_\nu\| \mathbb{B}_n.$$

By (6.2), one ends up with

$$y_\nu \in Q_\nu + M_0 \|x_\nu - \hat{x}_\nu\| \mathbb{B}_n.$$

Now, it suffices to pass to the limit with respect to $\nu \in N_2$ in order to see that $y \in Q$. We have proven in this way that $F(Q) \subseteq Q$ as required. □

Remark 6.4. Theorem 6.3 is a substantial improvement with respect to [13, Theorem 5.1]. Not only we are giving here a shorter and simpler proof, but we are also dispensing with the uniform boundedness condition (1.4). Thanks to Theorem 5.3, this bothersome hypothesis is automatically integrated in the controllability assumption made on the lower limit F.

Corollary 6.5. *Let $\mathscr{F}(\mathbb{R}^n, \mathbb{R}^n)$ be equipped with the metric δ (or with any other metric that induces Painlevé–Kuratowski convergence). Then,*

$$\mathscr{F}_{\text{contr}}(\mathbb{R}^n, \mathbb{R}^n) = \{F \in \mathscr{F}(\mathbb{R}^n, \mathbb{R}^n) \mid F \text{ is controllable}\}$$

is an open set.

Proof. This is immediate from Theorem 6.3. □

The complement of $\mathcal{F}_{\text{contr}}(\mathbb{R}^n, \mathbb{R}^n)$ in $\mathcal{F}(\mathbb{R}^n, \mathbb{R}^n)$ is therefore a closed set. It is then natural to consider the coefficient

$$\rho(F) = \inf_{\substack{G \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^n) \\ G \text{ uncontrollable}}} \delta(G, F)$$

as a tool for measuring the degree of controllability of a given F. We leave as open the problem that consists in estimating $\rho(F)$ and finding an uncontrollable $G \in \mathcal{F}(\mathbb{R}^n, \mathbb{R}^n)$ lying at minimal distance from F. This would extend the work initiated in [8].

APPENDIX

The recent book by Smirnov [20] contains large portions devoted to the analysis of differential inclusions governed by convex processes. There are several reasons why this class of dynamical systems deserves close attention. It is not the purpose of our paper to focus excessively on this point, but we would like to take this opportunity to mention a couple of things that are not yet well integrated by the mathematical community at large:

1. Differential inclusions governed by convex processes arise in many areas of applied mathematics. For instance, the problem of controlling a linear system by using positive inputs has been recognized as an important one since the pioneering works by Brammer [6] and Korobov [11] (see also Son [21]). The model under consideration is

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t) \\ u(t) \in P, \end{cases} \tag{A.1}$$

where A, B are matrices of appropriate sizes, and P is a closed convex cone regarded as the set of "positive" elements of some underlying control space (typically, P is the positive orthant of a given Euclidean space, say $\mathbb{R}^m$). Notice that the control model (A.1) fits into the framework (6.1) with F being the convex process given by $F(x) = Ax + B(P)$.

2. More often than not, one has to do with a differential inclusion

$$\dot{z}(t) \in \Phi(z(t)) \quad \text{a.e. on } [0, T] \tag{A.2}$$

whose right-hand side $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is not a convex process, not even a process for that matter. Typically, Φ is a multivalued map with closed graph and enjoying some kind of Lipschitz behavior. Linearizing (A.2) around a reference or equilibrium point $(\bar{x}, \bar{y}) \in \operatorname{gr} \Phi$, as done in the classical setting of single-value dynamical systems, is out the question here. Because a linear model would poorly reflect the complexity of (A.2), a more reasonable strategy consists in "convexifying" around $(\bar{x}, \bar{y})$. This essentially means changing Φ by a closed convex process $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ whose graph

$$\operatorname{gr} F = T_{\operatorname{gr}\Phi}(\bar{x}, \bar{y})$$

is the Clarke tangent cone to $\operatorname{gr}\Phi$ at $(\bar{x}, \bar{y})$ (cf. [3, Definition 5.2.1]). Loosely speaking, F can be seen as the "derivative" at $(\bar{x}, \bar{y})$ of the map Φ. This convexification mechanism works fine in many situations. For instance, Frankowska [7] succeded in deriving local controllability results for the differential inclusion (A.2) from (global) controllability of the convexified dynamical system (6.1).

3. Last but not the least, the model (6.1) admits a discrete counterpart that reads as follows:

$$\varrho_{k+1} \in F(\varrho_k) \quad \text{for } k = 0, 1, \ldots \tag{A.3}$$

Discrete iteration systems governed by convex processes have found many applications as well, see the book by Phat [17] for applications in control theory and the book by Makarov and Rubinov [15] for applications in mathematical economics. The fact that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a strict closed convex process has a surprising and welcome consequence: if a given trajectory $\vec{\varrho} = \{\varrho_k\}_{k \geq 0}$ of the discrete model (A.3) doesn't grow too fast in the sense that

$$\sum_{k=0}^{\infty} \frac{T^k}{k!} \|\varrho_k\| < \infty,$$

then such trajectory yields a solution to the continuous time model (6.1) by setting

$$z_{\vec{\varrho}}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \varrho_k.$$

As shown by Alvarez et al. [1], the function $z_{\vec{\varrho}} : [0, T] \to \mathbb{R}^n$ turns out to be a smooth solution to (6.1). This constructive result pleads again in favor of adopting the convex processes as key ingredients of the theory of differential inclusions.

Coming back to the main topic of our work, we would like to mention that the robustness result stated in Theorem 6.3 is just one of the many possible applications of the uniform boundedness principle established in Theorem 5.3. We could have derived also a robustness result concerning the controllability of a second-order differential inclusion, say

$$D\dot{z}(t) + M\ddot{z}(t) \in F(z(t)),$$

with D, M being matrices of size $n \times n$ and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ being a closed convex process. Second-order differential inclusions of this type serve for instance to model a mechanical system [9]

$$Kz(t) + D\dot{z}(t) + M\ddot{z}(t) = Bu(t)$$
$$u(t) \in P$$

controlled by means of a "positive" input function.

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