Analysis of a stabilized finite element method for fluid flows through a porous interface

Alfonso Caiazzo\textsuperscript{a,b}, Miguel A. Fernández\textsuperscript{b}, Vincent Martin\textsuperscript{c,b}
\textsuperscript{a}Weierstrass Institute, Mohrenstr. 39, 10117 Berlin, Germany
\textsuperscript{b}INRIA Paris-Rocquencourt, BP 105, 78153 Le Chesnay Cedex, France
\textsuperscript{c}UTC, LMAC, Royallieu, BP 20529, 60205 Compiègne, France

Abstract

This work is devoted to the numerical simulation of an incompressible fluid through a porous interface, modeled as a macroscopic resistive interface term in the Stokes equations. We improve the results reported in \textit{M2AN 42(6):961–990, 2008}, by showing that the standard Pressure Stabilized Petrov-Galerkin (PSPG) finite element method is stable, and optimally convergent, without the need for controlling the pressure jump across the interface.

Keywords: Stokes equation, porous interface, stabilized finite element method

1. Introduction

We consider a regular domain $\Omega \subseteq \mathbb{R}^d$, $d = 2$ or $3$, and a porous interface defined by a hyperplane domain $\Gamma \subseteq \mathbb{R}^{d-1}$, dividing $\Omega$ in two connected subdomains as $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$. We denote by $n_1$, $n_2$ the outgoing normals from each subdomain $\Omega_i$ at the interface, with $n_1 = -n_2$, and we define $n = n_1$. The fluid velocity $u$ and pressure $p$ are governed by the following modified Stokes equations \cite{1}:

$$\nabla p - \mu \Delta u + r_{\Gamma}\delta_{\Gamma} u = f \quad \text{in} \quad \Omega,$$

$$\text{div} \ u = 0 \quad \text{in} \quad \Omega,$$  \hspace{1cm} (1)

with a homogeneous Dirichlet condition on $\partial \Omega$. In (1), the symbol $\mu$ stands for the fluid viscosity, $f$ for a given volume force, $\delta_{\Gamma}$ for the Dirac measure on $\Gamma$, and $r_{\Gamma} > 0$ is a given interface resistance, related to the permeability and porosity of the interface. Without loss of generality, $r_{\Gamma}$ is assumed to be a constant scalar. For the sake of conciseness we limit ourselves to this problem. Nevertheless, the analysis below could be generalized to other problems involving pressure discontinuities, such as two-phase flows.

Problem (1) can be reformulated equivalently as a two-domain Stokes problem, complemented with the interface conditions

$$[u] = 0, \quad [\mu \nabla u \cdot n - pn] = -r_{\Gamma} u \quad \text{on} \quad \Gamma,$$ \hspace{1cm} (2)
where \([q] = q_1|_\Gamma - q_2|_\Gamma\) denotes the jump across \(\Gamma\) and \(q_i = q|_{\Omega_i}\) \((i = 1, 2)\).

In [1], problem (1) was discretized with an extension of the PSPG stabilized method (see [2]): an additional consistent term (based on (2)) was introduced to control the interface pressure jump. Numerical evidence showed, however, that this term did not improve noticeably the behavior of the numerical solution with respect to a standard PSPG stabilized formulation [1]. The aim of this note is to show that, indeed, stability and optimal accuracy can be derived without the need for this extra interface stabilization term (which is convenient in practice).

2. Finite element formulation

Let \(\{T_h\}_{0<h\leq 1}\) be a regular family of quasi-uniform triangulations of \(\Omega\), conforming with the interface \(\Gamma\). The corresponding triangulation of the interface is denoted by \(G_h\) and we set \(h = \max_{T\in T_h} h_T\), where \(h_T\) is the diameter of the element \(T\). We introduce the spaces \(V^0 = [H^0(\Omega)]^d\), \(Q = L^2(\Omega)\), and the finite element spaces of degree \(k \geq 1\), \(V^k_h\) and \(N^k_h\), equal order approximations of \(V\) and \(Q\):

\[
V^k_h = \{ v_h \in (C^0(\Omega))^d \mid v_{h|T} \in (P_k)^d \ \forall T \in T_h \} \cap V, \\
N^k_h = \{ q_{h|\Omega_i} \in C^0(\Omega_i), \ i = 1, 2 \mid q_{h|T} \in P_k \ \forall T \in T_h \} \cap Q.
\] (3)

Note that the space \(N^k_h\) of discrete pressures allows discontinuity at the interface \(\Gamma\). As underlined in [1], this is of utmost importance to get a correct approximation of the solution without excessive mesh refinement. Additionally, we introduce the spaces \(V^0_0 = \{ v \in V \mid v|_\Gamma = 0 \}\) and \(V^k_{0,h} = V_0 \cap V^k_h\).

Let us consider the two following bilinear forms

\[
A^r_\delta(x_h, y_h) \overset{\text{def}}{=} (\mu \nabla u_h, \nabla v_h) - (p_h, \text{div} v_h) + (r_\Gamma u_h, v_h)_\Gamma + (\text{div} u_h, q_h)
+ \delta \sum_{T \in T_h} \frac{h_T^2}{\mu} (\mu \Delta u_h + \nabla p_h, \nabla q_h)_T,
\]

\[
B^r_\delta(x_h, y_h) \overset{\text{def}}{=} A^r_\delta(x_h, y_h) - \delta \sum_{E \in G_h} \frac{h_E}{\mu} (\{\mu \nabla u_h \cdot n - p_h n]\ + r_\Gamma u_h, [q_h n])_E
\]

for all \(x_h = (u_h, p_h)\) and \(y_h = (v_h, q_h)\) in \(V^k_h \times N^k_h\) and \(\delta > 0\) is a stabilization parameter. The discrete formulation proposed and analyzed in [1] is based on \(B^r_\delta\). In this note, we consider the numerical analysis of the standard PSPG formulation

\[
A^r_\delta(x_h, y_h) = (f, v_h) + \delta \sum_{T \in T_h} \frac{h_T^2}{\mu} (f, \nabla q_h)_T \ \forall y_h \in V^k_h \times N^k_h. \tag{4}
\]
3. Stability analysis

Let us consider the mesh-dependent energy norm
\[ \|(u_h, p_h)\|_h^2 \overset{\text{def}}{=} \mu \|\nabla u_h\|_{0,\Omega}^2 + r_T \|u_h\|_{0,T}^2 + \delta \sum_{T \in \mathcal{T}_h} \frac{h_T^2}{\mu} \|\nabla p_h\|_{0,T}^2 + \frac{1}{\mu} \|p_h\|_{0,\Omega}^2. \]

Note that, unlike in [1], this norm provides no control on the interface pressure jump. We address now the stability of (4) in the \( \| \cdot \|_h \) norm.

By applying the inverse inequality (see [3])
\[ \|\Delta v_h\|_{0,T} \leq c\Delta h^{-1} \|\nabla v_h\|_{0,T}, \quad v_h \in V_h, \]
and the Schwarz and Young inequalities to the term \( \sum_{T \in \mathcal{T}_h} h_T^2 (\Delta u_h, \nabla p_h)_T \), we get the following coercivity estimate.

**Proposition 3.1.** Let \( \delta \) be such that \( 0 < \delta c_A^2 \leq 1 \). Then
\[ \mathcal{A}_\delta^T (x_h, x_h) \geq \frac{\mu}{2} \|\nabla u_h\|_{0,\Omega}^2 + r_T \|u_h\|_{0,T}^2 + \frac{\xi^2}{2} \geq \frac{1}{2} \left( \|(u_h, p_h)\|_h^2 - \frac{1}{\mu} \|p_h\|_{0,\Omega}^2 \right) \]
for all \( x_h = (u_h, p_h) \in V_h^k \times Q_h^k \), with \( \xi^2 \overset{\text{def}}{=} \delta \sum_{T \in \mathcal{T}_h} \frac{h_T^2}{\mu} \|\nabla p_h\|_{0,T}^2. \)

The stability and the optimal convergence are stated in the following result.

**Proposition 3.2.** Under the assumption of Proposition 3.1 there holds:
(i) there exists a constant \( \beta = \beta(\delta, \frac{r_T}{r}) \) independent of \( h \), such that
\[ \inf_{x_h \in V_h^k \times Q_h^k} \sup_{y_h \in V_h^k \times Q_h^k} \frac{\mathcal{A}_\delta^T (x_h, y_h)}{\|x_h\|_h \|y_h\|_h} \geq \beta. \]
Moreover, if \( \delta \ll 1 \) we have \( \beta \sim \delta \), and \( \beta = O(\mu/r_T) \) for \( r_T/\mu \gg 1 \);
(ii) let \( (u_h, p_h) \) be the solution of (4) and assume that \( (u, p) \), the solution of (1), is such that \( u_i \in [H^{k+1}(\Omega_i)]^d, p_i \in H^k(\Omega_i), i = 1, 2 \). There holds
\[ \|(u - u_h, p - p_h)\|_h \leq c(\beta^{-1}) h^k \sum_{i=1,2} \left[ \left( 1 + r_T^{\frac{1}{2}} h_T^{-\frac{1}{2}} \mu^{-\frac{1}{2}} + \delta^{-\frac{1}{2}} \right) \mu^{-\frac{1}{2}} \|u\|_{k+1,\Omega_i} \right. \]
\[ + \left. \left( 1 + \delta^{-\frac{1}{2}} \right) \mu^{-\frac{1}{2}} \|p\|_{k,\Omega_i} \right), \]
where \( c \) is a positive constant, independent of \( h \), that behaves as \( 1/\beta \).

We remark that the stability and convergence results are essentially the same as the ones given in [1], but without the need for the extra stabilization term. Note that the scaling \( \sqrt{r_T/\mu} \) is present in both cases. The \( \inf\sup \) constant \( \beta \) and the estimate constant \( c \) have also the same asymptotic behavior as in [1].
Proof. For the sake of conciseness, we prove only point (i). The proof of (ii) follows [1], owing to the stability of $A^r_\delta$. Let $x_h = (u_h, p_h) \in V^k_h \times N^k_h$. Given (5), the inf-sup stability of $A^r_\delta$ requires additional stability estimates needed to control the pressure.

A pressure $p \in L^2_0(\Omega)$ has zero mean in $\Omega$, but this is not true in general for its restriction to $\Omega_i$, $i = 1, 2$. Following an argument of [4], we decompose $p_h \in N^k_h \subset L^2_0(\Omega_i)$ as $p_h = p^0_h + \overline{p}_h$, with $p^0_{h,i} \in L^2_0(\Omega_i)$ and $\overline{p}_{h,i} \overset{\text{def}}{=} \frac{(p_h - 1)_{\Omega_i}}{\mu_{\Omega_i}}$ (i.e., $p^0_h$ has zero mean over each subdomain and $\overline{p}_h$ is constant in each subdomain). The following relations hold:

$$
\|p_h\|_{0,\Omega}^2 = \|p^0_h\|_{0,\Omega}^2 + \|\overline{p}_h\|_{0,\Omega}^2, \quad \overline{p}_{h,1}|\Omega_1| + \overline{p}_{h,2}|\Omega_2| = 0, \quad \|\overline{p}_h\|_{0,\Omega}^2 = \overline{p}_{h,1}|\Omega_1| + \overline{p}_{h,2}|\Omega_2|.
$$

(8)

We show how to control separately $p^0_h$ and $\overline{p}_h$. Since $p^0_{h,i} \in L^2_0(\Omega_i)$, $i = 1, 2$, there exists a function $v^0 \in V_0$ such that $v^0_i \in [H^1_0(\Omega_i)]^d$, $-\text{div} \, v^0 = \frac{1}{\mu} p^0_{h,i}$ and $\|v^0\|_{1,\Omega} \leq c_o \|p^0_h\|_{0,\Omega}$. We take $v^0_h$ as the Scott-Zhang interpolant of $v^0$ into $V_{h,h}$, defined separately on each subdomain. Using the properties of the Scott-Zhang operator [3], we also have $\|v^0_h\|_{1,\Omega} \leq c'_o \|p^0_h\|_{0,\Omega}$ and $\|\overline{p}_h\|_{0,\Omega} \leq c_o h \|v^0\|_{1,\Omega}$. Since $v^0_i, v^0_{i,h} \in [H^1_0(\Omega_i)]^d$, $i = 1, 2$, and $\overline{p}_h$ is constant on each subdomain, we have $(v^0 - v^0_h)|_\Gamma = 0$, $(\overline{p}_h, \text{div} \, v^0_h) = 0$ and $(\overline{p}_h, \text{div} \, v^0) = 0$. Hence, using the fact that $p_h \in N^k_h$ is continuous in $\Omega_1$ and $\Omega_2$ we obtain, integrating by parts in each subdomain:

$$
-(p_h, \text{div} \, v^0_h) = -(p^0_h, \text{div} \, v^0) + (p^0_h, \text{div} (v^0 - v^0_h)) \geq \frac{1}{\mu} \|p^0_h\|_{0,\Omega}^2 - \xi \|\overline{p}_h\|_{0,\Omega},
$$

with $\xi$ defined in Proposition 3.1. Using Young’s inequality, this yields

$$
A^r_\delta (x_h, (v^0_h, 0)) \geq -c'_o \mu \|\nabla u_h\|_{0,\Omega} \left( \frac{1}{\mu} \|p^0_h\|_{0,\Omega} - \delta^{-\frac{1}{2}} c_o \xi \|\overline{p}_h\|_{0,\Omega} \right) \left( \frac{1}{\mu} \|p^0_h\|_{0,\Omega} + \frac{1}{\mu} \|p_h^0\|_{2,\Omega} \right) \geq \frac{1}{2} \|p_h^0\|_{0,\Omega}^2 - (c'_o \mu)^2 \|\nabla u_h\|_{0,\Omega}^2 - \frac{c_o^2 \xi^2}{\delta^2}.
$$

(9)

To handle the constant part of the pressure, we need the following Lemma (whose proof can be found in [4] in a more complex case):

Lemma 3.1. There exist two (non-constant) functions $\overline{v}_h^\alpha \in V^1_h$, $\alpha = 1, 2$, defined over the whole domain $\Omega$ such that $\int_\Gamma \overline{v}_h^\alpha \cdot n_1 = -\int_\Gamma \overline{v}_h^\alpha \cdot n_2 = \frac{\overline{p}_h, |\Omega_\alpha|}{\mu}$ and

$$
\mu \|\nabla \overline{v}_h^\alpha\|_{0,\Omega} \leq \overline{c} \mu^{-\frac{1}{2}} \|\overline{p}_h, |\Omega_\alpha|, \quad \frac{1}{\mu} \|\overline{v}_h^\alpha\|_{0,\Omega} \leq \overline{c} \mu^{\frac{1}{2}} \|\overline{p}_h, |\Omega_\alpha|.
$$

4
Let $\mathbf{v}_h \mathbf{\defeq} \mathbf{v}_h - \mathbf{v} \in \mathbf{V}_h$. Since $\nabla \mathbf{p}, 0$ and using (8) and Lemma 3.1, we have

$$-(p_h, \text{div} \mathbf{v}_h) = \sum_{i=1,2} (\mathbf{p}_h, i, (\mathbf{v}_h - \mathbf{v}_h) \cdot n_i)_r$$

$$= \mu^{-1} (p_{h,1}^2 |\Omega_1| - \mathbf{p}_{h,1} \cdot \mathbf{p}_{h,2} (|\Omega_2| + |\Omega_1|) + p_{h,2}^2 |\Omega_2|)$$

$$= 2 \mu^{-1} (p_{h,1}^2 |\Omega_1| + p_{h,2}^2 |\Omega_2|) = 2 \mu^{-1} \|p_h\|_{0,\Omega},$$

and, by applying Lemma 3.1 once more, and using the fact that $\|p\|_{0,\Omega}$, there follows

$$-(p_h, \text{div} \mathbf{v}_h) = -(p_h, \text{div} \mathbf{v}_h) - (p_h, \text{div} \mathbf{v}_h)$$

$$\geq 2 \mu^{-1} \|p_h\|_{0,\Omega}^2 - \|p_h\|_{0,\Omega}^2 + \frac{1}{4 \mu} \left(\|p_{h,1}\|_{0,\Omega} + \|p_{h,2}\|_{0,\Omega}\right)^2$$

$$\geq \frac{1}{\mu} \|p_h\|_{0,\Omega}^2 - \frac{d^2}{\mu} \|p_h\|_{0,\Omega}^2,$$

where we recall that $d$ denotes the spatial dimension. Hence,

$$A_h^{\Gamma} (x_h, (\mathbf{v}_h, 0)) \geq -2 c \mu \|\mathbf{v}_h\|_{0,\Omega} \mu^{-\frac{1}{2}} \|\mathbf{p}_h\|_{0,\Omega} - 2C r_1 \mu^{-\frac{1}{2}} \|\mathbf{u}_h\|_{0,\Gamma} \mu^{-\frac{1}{2}} \|\mathbf{p}_h\|_{0,\Omega}$$

$$\geq \frac{1}{\mu} \|p_h\|_{0,\Omega}^2 - \frac{d^2}{\mu} \|p_h\|_{0,\Omega}^2 - \frac{4 \epsilon}{\mu} \|\mathbf{u}_h\|_{0,\Omega}^2 - 4C^2 r_1 \mu^{-\frac{1}{2}} \|\mathbf{u}_h\|_{0,\Gamma}.$$  

(10)

Therefore, by taking $y_h = (\lambda \mathbf{v}_h + (1 - \lambda) \mathbf{v}_h, 0)$, with $\lambda = \frac{1 + 2d^2}{2(1 + d^2)} \in (0, 1)$, and using (9) and (10), we obtain

$$A_h^{\Gamma} (x_h, y_h) \geq \left(\frac{\lambda}{2} - (1 - \lambda) d c^2 \right) \frac{1}{\mu} \|p_h\|_{0,\Omega}^2 + \frac{\lambda}{2} \frac{1}{\mu} \|p_h\|_{0,\Omega}^2$$

$$- \left(\lambda (c_r^2 + (1 - \lambda) 4 d^2) \mu \|\nabla \mathbf{u}_h\|_{0,\Omega}^2 - \lambda \frac{c^2}{\delta} \mathbf{\xi}_i^2 - (1 - \lambda) 4C^2 r^2 \mu \|\mathbf{u}_h\|_{0,\Gamma}^2 \right)$$

$$\geq \frac{1}{4 \epsilon c \mu} \|p_h\|_{0,\Omega}^2 - c_{\max}^2 \left(\left\|\mathbf{u}_h, p_h\right\|_{0,\Omega}^2 - \frac{1}{\mu} \|p_h\|_{0,\Omega}^2\right),$$  

(11)

where we have introduced $c \mathbf{\defeq} 1 + d c^2$, and $c_{\max}^2 = \max\left\{ \left(c_r^2 + 4 d^2\right), \frac{1}{7} c^2, \frac{c^2}{\delta}, 4C^2 \mu \right\}$. Note that, unlike the other constants that are dimensionless, $c^2$ has the dimension of the inverse of a distance. Equation (11) provides a control on the pressure.
To conclude the proof, we take a test function $z_h \overset{\text{def}}{=} (1 - \omega)x_h + \omega y_h$, with $\omega \overset{\text{def}}{=} \frac{2\tilde{c}}{1 + 2\tilde{c}(1 + 2c_{\text{max}})} \in (0,1)$, and apply (5) and (11), to obtain
\[
\mathcal{A}_\delta^\Gamma(x_h, z_h) \geq \frac{1}{2}(1 - \omega) \left( \| (u_h, p_h) \|_h^2 - \frac{1}{\mu} \| p_h \|_{0,\Omega}^2 \right) \\
+ \omega \left( \frac{1}{4c\mu} \| p_h \|_{0,\Omega}^2 - c_{\text{max}}^2 \left( \| (u_h, p_h) \|_h^2 - \frac{1}{\mu} \| p_h \|_{0,\Omega}^2 \right) \right) \\
\geq \left( \frac{1 - \omega}{2} - \omega c_{\text{max}}^2 \right) \left( \mu \| \nabla u_h \|_{0,\Omega}^2 + r_\Gamma \| u_h \|_{0,\Gamma}^2 + \xi^2 \right) + \frac{\omega}{4c\mu} \| p_h \|_{0,\Omega}^2 \\
\geq \frac{1}{2 \left( 1 + 2\tilde{c}(1 + 2c_{\text{max}}) \right) } \| x_h \|_h^2.
\]

Moreover, it can be shown that $z_h$ can be controlled by $x_h$ as
\[
\| z_h \|_h \leq (1 - \omega) \| x_h \|_h + \omega \left( \| (v_0^1, 0) \|_h + \| (\bar{v}_h^1, 0) \|_h + \| (\bar{v}_h^0, 0) \|_h \right) \\
\leq (1 - \omega) \| x_h \|_h + \omega c_{\text{max}} \mu^{-\frac{1}{2}} \| p_h \|_{0,\Omega} + \omega \sqrt{2}(\tilde{c}^2 + \frac{r_\Gamma c^2}{\mu}) \mu^{-\frac{1}{2}} \| p_h \|_{0,\Omega} \\
\leq (1 - \omega) \| x_h \|_h + \omega \sqrt{2} c_{\text{max}} \| x_h \|_h = \left( 1 - \omega + \omega \sqrt{2} c_{\text{max}} \right) \| x_h \|_h \\
\leq \frac{1 + 2\tilde{c} c_{\text{max}}(2c_{\text{max}} + \sqrt{2})}{1 + 2\tilde{c}(1 + 2c_{\text{max}})} \| x_h \|_h.
\]

Combining (12) and (13) we obtain that the global inf-sup condition (6) follows with a constant $\beta \overset{\text{def}}{=} \left[ 2(1 + 2\tilde{c} c_{\text{max}}(2c_{\text{max}} + \sqrt{2}) \right]^{-1}$. The stated asymptotic behavior of $\beta$ follows from the definition of $c_{\text{max}}$. □


