ASYMPTOTICS FOR AT THE MONEY LOCAL VOL BASKET OPTIONS

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To the memory of Peter Laurence, who passed away unexpectedly during the final stage of the preparation of this manuscript.

Abstract. We consider a basket or spread option on based on a multi-dimensional local volatility model. Bayer and Laurence [Comm. Pure. Appl. Math., to appear] derived highly accurate analytic formulas for prices and implied volatilities of such options when the options are not at the money. We now extend these results to the ATM case. Moreover, we also derive similar formulas for the local volatility of the basket.

1. Introduction

For a local volatility type model for a basket of stocks, whose forward prices are given by

$$dF_i(t) = \sigma_i(F_i(t))dW_i(t), \quad i = 1, \dots, n, \tag{1.1}$$

$$d\left\langle W_i\,,\, W_j\right\rangle(t) = \rho_{ij}dt, \quad i, j = 1, \dots, n, \tag{1.2}$$

with a given correlation matrix ρ, we consider basket options with a payoff

$$P(\mathbf{F}) = \left(\sum_{i=1}^{n} w_i F_i - K\right)^+,$$

where we generally denote in bold face a vector of the corresponding italic components, as in $\mathbf{F} = (F_1, \dots, F_n)$. Since we only assume that at least one of the weights $w_1, \dots, w_n$ is positive, we will refer to options of that type as *generalized spread options*.

The purpose of this paper is to provide an explicit first order accurate short time expansion of the price $C_{\mathcal{B}}(\mathbf{F}_0, K, T)$ of the above option using the heat kernel expansion technique (see, for instance, [11], [10], [18]) when the option is at the money. Moreover, from the asymptotic formula for the option price we also obtain an asymptotic formula for the implied and for the local volatility.[1] Thereby we complement the results obtained in [4], where a first order accurate asymptotic formula was given when the option is *not* at the money. (The zero order accurate

2010 *Mathematics Subject Classification.* Primary 91G60; Secondary 91G20, 58J90.

Key words and phrases. Basket options, spread options, implied volatility, asymptotic formulas, heat kernel expansion.

[1] Since we consider spread options here (for which $\sum_i w_i F_{0,i}$ may be negative), we derive implied volatilities both in the Black-Scholes and in the Bachelier sense.

formula is well-known, see, for instance [1]. When the option is not at the money, alternative first order accurate results can be found in [10].) Such asymptotic formulas are highly relevant, in particular when the dimension of the model is high (say $n > 3$), since then traditional (simulation or PDE) techniques to compute $C_{\mathcal{B}}$ fail or are at least very time consuming. In fact, for a wide range of different parameters, [4] show numerically that their asymptotic formula is remarkably close to the true price as given by the model, even for not so small maturities T (like 5 or even 10 years), for dimensions of up to $n = 100$ (or even more). The same holds true when the option is at the money, see Section 6.

We now sketch the procedure for deriving the asymptotic formulas, highlighting the differences to the non-ATM case.

- In the first step, we derive a Carr-Jarrow formula for the basket option price, separating the price into the intrinsic value of the option (which vanishes in the ATM case) and an integral over the arrival manifold $\{\sum_i w_i F_i = K\}$ with respect to the transition density $p(\mathbf{F}_0, \mathbf{F}, T)$. This is done in Section 2.
- The first terms in the heat kernel expansion of $p(\mathbf{F}_0, \mathbf{F}, T)$ are computed. In the non-ATM case, we a zero-order heat kernel expansion was sufficient to get first order accurate formulas for the implied volatilities. At the money, we actually need to add one additional term in the heat kernel expansion. The heat kernel coefficients are computed in Lemma 3.4.
- The afore-mentioned integral on the arrival manifold is essentially an integral with respect to the rapidly decaying kernel $\exp\left(-d(\mathbf{F}_0, \mathbf{F})^2/(2T)\right)$, where d denotes the Riemannian (geodesic) distance induced by the stock price process. Hence, the integral can be approximated using Laplace's expansion for $T \to 0$, which involves the minimizer $\mathbf{F}^*$ of $\mathbf{F} \mapsto d(\mathbf{F}_0, \mathbf{F})^2$ subject to $\sum_i w_i F_i = K$. In the general case, this minimizer has to be computed numerically, while it is obviously given by $\mathbf{F}^* = \mathbf{F}_0$ when the option is at the money. On the other hand, the formulas are much longer and more complex due to the higher order heat kernel expansion used, see Proposition 3.3 together with Lemma 3.2 and 3.5.
- In Section 4, we use the same Laplace's expansion technique to derive the local volatility of the basket, see Proposition 4.1.
- Finally, in Section 5, an asymptotic expansion for the implied volatilities is computed by a comparison of coefficients between the asymptotic expansion of the basket price derived in Proposition 5.1 and asymptotic expansions of the Black-Scholes and Bachelier formulas, respectively, see equation (5.2), (5.3) and (5.4).

An alternative way to derive the asymptotic expansion for at-the-money options would be to start from the non-at-the-money formulas and pass to the limit. This would involve un-determined terms "$\frac{0}{0}$", which would need to be resolved by the l'Hopital rule. In particular, we would have to compute limits of derivatives of the optimal configuration, which is not known in closed form when the option is not at-the-money. Still, one could follow that approach using similar techniques as

in [3], but the derivation would hardly be any simpler than directly starting from scratch again (the course of action chosen in this article).

In Section 6 we present numerical examples for one particular choice of a local volatility model, namely the CEV model, corresponding to $\sigma_i(F_i) = \xi_i F_i^{\beta_i}$, $0 \le \beta_i \le 1$, $1 \le i \le n$. The numerical observations supports the claimed accuracy of the asymptotic price formulas. In fact, comparisons with highly accurate reference solutions show that the asymptotic formulas indeed have the suggested rates of convergence as $T \to 0$. Even more, they indicate that the formulas, in particular the first order formula, are highly accurate even for large maturities such as $T = 10$ years, thereby confirming the observations in [4]. Note, however, that our derivation is not fully rigorous, as we assume that the initial price $\mathbf{F}_0$ is far enough from the boundary of $\mathbb{R}^n_+$ such that boundary effects (caused by singularities of σ_i at $F_i = 0$) are not felt. As verified by the numerical experiments, the assumption seems justified (at least in an equity setting). We refer to the discussion in [4, Section 4] on how to verify these assumptions. See also [2] and [6], [7] for related problems.

2. Basket Carr-Jarrow formula

Consider a basket $\mathcal{B} = \sum w_i F_i$ with weights $w_i \in \mathbb{R}$. Following [5] and [4], we are now going to derive a Carr-Jarrow formula for the price of a generalized spread option on the basket, i.e., a decomposition of the price of the option into the intrinsic value and an integral over the arrival manifold $\{\mathcal{B} = K\}$. Take the Itô derivative of the basket's price:

$$\begin{aligned} d\sum_{i=1}^n w_i F_i(t) &= \sum_{i=1}^n w_i \sigma_i(F_i(t)) dW_i(t) \\ &= \sqrt{\underbrace{\sum_{i,j=1}^n w_i w_j \sigma_i(F_i(t)\sigma_j(F_j(t))\rho_{ij}}_{\sigma^2_{N,\mathcal{B}}}}\, d\bar{W}(t), \end{aligned}$$

for a new Brownian motion $\bar{W}$. Here we have used the notation $\sigma_{N,B}$ to indicate the "normal volatility" of the basket which must not be confused with the lognormal (Black) volatility $\sigma_{\mathcal{B}} = \frac{\sigma_{N,\mathcal{B}}}{\sum_{i=1}^n w_i F_i}$ used in reference [1]. Therefore, by the Itô-Tanaka formula we have

$$d\left(\sum_{i=1}^n w_i F_i(t) - K\right)^+ = \sum_{i=1}^n w_i \mathbf{1}_{\sum w_i F_i(t) > K}\, dF_i(t) + \frac{1}{2}\delta_{\{\mathbf{F}:\sum w_i F_i(t) = K\}}\, \sigma^2_{N,\mathcal{B}}(\mathbf{F}(t))dt.$$

Integrating we obtain

$$\left(\sum_{i=1}^{n} w_i F_i(T) - K\right)^+ = \left(\sum_{i=1}^{n} w_i F_i(0) - K\right)^+ + \sum_{i=1}^{n} w_i \int_0^T \mathbf{1}_{\sum w_i F_i(u) > K}\, dF_i(u) + \\ + \frac{1}{2}\int_0^T \delta_{\{\mathbf{F}(u):\sum w_i F_i(u) = K\}}\, \sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}(u))du.$$

Letting $\mathcal{E}_K = \{\mathbf{F} \in \mathbb{R}^n_+ : \sum w_i F_i = K\}$ and taking conditional expectations with respect to the filtration $\mathcal{F}_0$ at time 0, we obtain, assuming $F_i(t)$ is a martingale for each i[2]:

$$C_{\mathcal{B}}(\mathbf{F}_0, K, T) = \left(\sum_{i=1}^{n} w_i F_i(0) - K\right)^+ + \frac{1}{2}\int_0^T E\left[\sigma^2_{\mathcal{N},\mathcal{B}}\, \delta_{\mathcal{E}_K}(\mathcal{B}_t)\right] dt.$$

Letting $|\mathbf{w}| = \sqrt{\sum_{i=1}^{n} w_i^2}$, and denoting by H_{n-1} the Hausdorff measure, which on the hyperplane $\mathcal{E}_K$ coincides with Lebesgue measure, a simple use of the co-area formula (see [8]), and using that $|\nabla(\sum_{i=1}^{n} w_i F_i)| = |w|$, we see that the expectation, when expressed in terms of the joint transition density, is given by:

$$C_{\mathcal{B}}(\mathbf{F}_0, \mathbf{K}, T) = \left(\sum_{i=1}^{n} w_i F_i(0) - K\right)^+ + \\ + \frac{1}{2}\int_0^T \frac{1}{|\mathbf{w}|}\int_{\mathcal{E}_K} \sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}) p(\mathbf{F}_0, \mathbf{F}, u) H_{n-1}(d\mathbf{F})du.$$

Therefore, we arrive at the proposition:

Proposition 2.1. *The value of a call option on a basket $\mathcal{B}$ is given by*

$$C_{\mathcal{B}}(\mathbf{F}_0, K, T) = \left(\sum_{i=1}^{n} w_i F_i(0) - K\right)^+ + \\ + \frac{1}{2}\int_0^T \frac{1}{|\mathbf{w}|}\int_{\mathcal{E}_K} \sum_{i,j=1}^{n} w_i w_j \sigma_i(F_i)\sigma_j(F_j)\rho_{ij} p(\mathbf{F}_0, \mathbf{F}, u) H_{n-1}(d\mathbf{F})du. \quad (2.1)$$

Using the formula for the basket's local volatility, [1] or [10], expressed in the notation introduced above, after canceling common factors we also have the

[2] In many cases of interest, $F_i(t)$ is only a local martingale and not a martingale. But the discrepancy is not "felt" for short times, since the set of paths that can reach the boundary have small probability, in this limit. This is known as the principle of "not feeling the boundary" for small times and is born out by our numerical results. More surprisingly the boundary is not felt, even for quite large times.

Proposition 2.2. *The local volatility of the basket option is given by:*

$$\sigma_{loc}^2(K,T)K^2 = \frac{\int\limits_{\mathcal{E}(K)} \sum\limits_{i,j=1}^{n} w_i w_j \sigma_i(F_i)\sigma_j(F_j)\rho_{ij} p(\mathbf{F}_0,\mathbf{F},T) H_{n-1}(d\mathbf{F})}{\int\limits_{\mathcal{E}(K)} p(\mathbf{F}_0,\mathbf{F},T) H_{n-1}(d\mathbf{F})}.$$

3. A general asymptotic expansion procedure

The starting point is the basket Carr-Jarrow formula derived above for the calculation of the option prices as in Proposition 2.1 and the Proposition 2.2 for the calculation of the local volatilities. The next step is to approximate the transition density there using the heat kernel. For reasons that will become clear in the course of the asymptotics, it will be necessary to use the so-called geometric expansion

$$p(\mathbf{F}_0,\mathbf{F},t) = \frac{1}{(2\pi T)^{\frac{n}{2}}} \sqrt{\det g(\mathbf{F})} e^{-\frac{d^2(\mathbf{F}_0,\mathbf{F})}{2t}} (u_0(\mathbf{F_0},\mathbf{F}) + t u_1(\mathbf{F_0},\mathbf{F})) + o(t). \tag{3.1}$$

For a detailed exposition of the geometrical underpinning of (3.1) we refer to [10], [18], [11], [15] and [4]. Here, we just give a very quick reminder. The state space $\mathbb{R}^n$ is equipped with a Riemannian metric by defining the inverse g^{-1} of the metric tensor by

$$g^{ij}(\mathbf{F}) = \sigma_i(F_i)\rho_{ij}\sigma_j(F_j), \quad 1 \le i,j \le n.$$

Hence, the metric tensor itself is given by

$$g_{ij}(\mathbf{F}) = \sigma_i(F_i)^{-1}\rho^{ij}\sigma_j(F_j)^{-1}, \quad 1 \le i,j \le n,$$

with determinant

$$\det g(\mathbf{F}) = \det\left(\rho^{-1}\right) \prod_{k=1}^{n} \sigma_k(F_k)^{-2}$$

(where ρ^{ij} denotes the (i,j)-component of the inverse matrix ρ^{-1} of the correlation matrix ρ). The (geodesic) distance between two points $\mathbf{F}_0$ and $\mathbf{F}$ is denoted by $d(\mathbf{F}_0,\mathbf{F})$.

The *specific form* of these quantities in the setting of local volatility models has no relevance in our initial asymptotic derivations, which can be obtained for generic versions of these. So, to lighten the notation and streamline the presentation, we first derive the asymptotic expansions without any specific reference to these and then plug in the specific form only at the end of the process in order to produce the required concrete asymptotic expansions.

Plugging the heat kernel expansion (3.1) into the expressions in Propositions 2.1 and 2.2, respectively, we see that we have to compute expressions of the form

$$\frac{1}{(2\pi t)^{n/2}} \int_{\mathcal{E}_K} \Psi(\mathbf{F}) \exp\left(-\frac{d(\mathbf{F}_0,\mathbf{F})^2}{2t}\right) H_{n-1}(d\mathbf{F}), \tag{3.2}$$

where

$$\Psi(\mathbf{F}) = \bar{u}_i(\mathbf{F}_0,\mathbf{F}) := \sqrt{\det g(\mathbf{F})}\sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}) u_i(\mathbf{F}_0,\mathbf{F}), \quad i = 0,1, \tag{3.3}$$

for the option price and for the numerator in Proposition 2.2 and

$$\Psi(\mathbf{F}) = \hat{u}_i(\mathbf{F}_0, \mathbf{F}) := \sqrt{\det g(\mathbf{F})} u_i(\mathbf{F}_0, \mathbf{F}), \quad i = 0, 1 \tag{3.4}$$

for the denominator in Proposition 2.2.

The integral on the $n-1$ dimensional subspace $\mathcal{E}_K$ of $\mathbb{R}^n$ can be transformed into an integral over $\mathbb{R}^{n-1}$, by eliminating one of the variables. We choose to eliminate the n-th one, using the payoff

$$F_n(F_1, \dots, F_{n-1}, K) = \frac{1}{w_n}\left(K - \sum_{i=1}^{n-1} w_i F_i\right), \tag{3.5}$$

Denoting

$$\mathbf{G} = (F_1, \dots, F_{n-1}) \in \mathbb{R}_+^{n-1},$$
$$\mathbf{G}_K = \left\{\mathbf{G} \in \mathbb{R}^{n-1} \ : \ \sum_{i=1}^{n-1} w_i F_i < K\right\},$$

so that for our hyperplane's intersection

$$\mathcal{E}_K \cap \mathbb{R}_+^n = \left\{\mathbf{F} \in \mathbb{R}_+^n \ : \ \mathbf{F} = \left(\mathbf{G}, \frac{1}{w_n}\left(K - \sum_{i=1}^{n-1} w_i F_i\right)\right), \mathbf{G} \in \mathbf{G}_K\right\}.$$

Note that the set $\mathbf{G}_K$ is introduced in order to ensure that F_n in (3.5) is non-negative, as it needs to be. The set $\mathcal{E}_K$ is an $n-1$ dimensional hyperplane in $\mathbb{R}_+^n$.

Note that, when we parametrize the hyperplane $\mathcal{E}_k$ using $(F_1, \dots, F_{n-1})$, as in (3.5)

$$F_K(F_1, \dots, F_{n-1}) = (F_1, \dots, F_{n-1}, F_n(F_1, \dots, F_{n-1}, K)),$$

we will always assume that the weight multiplying F_n is *positive*. This can always be achieved by choosing as the n-th asset one of the assets with a positive weight. Then for the surface measure, we have

$$dH_{n-1} = \sqrt{1 + |\nabla F_n|^2} dF_1 \dots dF_{n-1} = \frac{|\mathbf{w}|}{|w_n|} dF_1 \dots dF_n.$$

In this notation, with $\Lambda = \frac{d^2}{2}$, the integral (3.2) reads

$$\frac{1}{(2\pi t)^{n/2}} \frac{|\mathbf{w}|}{|w_n|} \int_{\mathbf{G}_K} e^{-\frac{\Lambda(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G}))}{t}} \Psi(\mathbf{F}_K(\mathbf{G})) dF_1 \dots dF_{n-1} =$$
$$\frac{1}{(2\pi t)^{n/2}} \frac{|\mathbf{w}|}{|w_n|} \int_{\mathbf{G}_K} e^{-\frac{\Phi(\mathbf{G})}{t}} \Psi(\mathbf{G}) d\mathbf{G}, \tag{3.6}$$

using the notation $\Phi(\mathbf{G}) := \Lambda(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G}))$ and (by abuse of notation) $\Psi(\mathbf{G}) := \Psi(\mathbf{F}_K(\mathbf{G}))$. We now use *Laplace asymptotics* for multiple integrals. The main contribution comes from a neighborhood of the minimum point.

$$\begin{aligned} \mathbf{G}^* &= \arg\min_{\mathbf{G} \in \mathbf{G}_K} d^2(\mathbf{F}_0, (\mathbf{G}, F_n(\mathbf{G}, K)), \\ &= d^2(\mathbf{F}_0, \mathcal{E}_K). \end{aligned} \tag{3.7}$$

Set $\mathbf{F}_K^* = (\mathbf{G}^*, F_n(\mathbf{G}^*, K))$. (Of course, when the option is at the money, we have $\mathbf{G}^* = (F_{0,1}, \dots, F_{0,n-1})$.)

Order zero. The zero-th order term in the Laplace expansion of

$$\int_{\mathbf{G}_K} e^{-\frac{\Phi(\mathbf{G})}{t}} \Psi(\mathbf{G}) d\mathbf{G}$$

is identical to the one in [4] except that in the present setting we have $d(\mathbf{F}_0, \mathbf{F}_K^*) = 0$. We get, as in [4]

$$t^{\frac{n-1}{2}} \Psi(\mathbf{G}^*) \times \int_{\mathbb{R}^{n-1}} e^{-\frac{z^T Q z}{2}} dz_2 \dots dz_n = t^{\frac{n-1}{2}} \Psi(\mathbf{G}^*) \frac{(2\pi)^{\frac{n-1}{2}}}{(\det Q)^{\frac{1}{2}}},$$

where $Q = D^2\Phi(\mathbf{G}^*)$ is the Hessian of Φ at the minimum point. Thus, bringing back the missing factor and taking into account that $\mathbf{F}_K^* = \mathbf{F}_0$ in the current (ATM) setting, we see that the lowest order term in the Laplace expansion of (3.2) is

$$h_0^\Psi := \frac{|\mathbf{w}|}{|w_n|} \frac{1}{\sqrt{2\pi t \det Q}} \Psi(\mathbf{F}_0). \tag{3.8}$$

Order one. For obtaining first order implied or local volatility terms in the ATM regime, we need to push the Laplace expansion one step further, i.e., we need one additional term for

$$\int_{\mathbf{G}_K} e^{-\frac{\Phi(\mathbf{G})}{t}} \Psi(\mathbf{G}) d\mathbf{G}$$

Hence, we apply the (multi-variate) Taylor expansion for $\Phi(\mathbf{G}) := \Lambda(\mathbf{F}_0, \mathbf{F}_K(\mathbf{G}))$ up to order 4 around the maximizer $\mathbf{G}^*$, which can be expressed in tensor notation as

$$\Phi(\mathbf{G}) = \Phi(\mathbf{G}^*) + \underbrace{D\Phi(\mathbf{G}^*)}_{=0}(\mathbf{G} - \mathbf{G}^*) + \frac{1}{2} D^2\Phi(\mathbf{G}^*)(\mathbf{G} - \mathbf{G}^*)^{\otimes 2} +$$
$$+ \frac{1}{6} D^3\Phi(\mathbf{G}^*)(\mathbf{G} - \mathbf{G}^*)^{\otimes 3} + \frac{1}{24} D^4\Phi(\mathbf{G}^*)(\mathbf{G} - \mathbf{G}^*)^{\otimes 4} + \cdots,$$

with

$$D^k\Phi(\mathbf{x})\mathbf{y}^{\otimes k} := \sum_{i_1,\dots,i_k} \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} \Phi(\mathbf{x}) y_{i_1} \cdots y_{i_k}.$$[3]

Of course, we are aware that when the option is at the money, the optimal configuration is the same as the initial configuration $\mathbf{F}_0$. Nonetheless, we think that using a different symbol for the optimal configuration at this stage leads to a clearer exposition of the underlying ideas. Likewise, we apply Taylor expansion up to second order for the map $\Psi(\mathbf{G})$ around $\mathbf{G}^*$,

$$\Psi(\mathbf{G}) = \Psi(\mathbf{G}^*) + D\Psi(\mathbf{G}^*)(\mathbf{G} - \mathbf{G}^*) + \frac{1}{2} D^2\Psi(\mathbf{G}^*)(\mathbf{G} - \mathbf{G}^*)^{\otimes 2} + \cdots.$$

[3] This notation makes sense as any multi-linear map on a vector space – such as $D^k\Phi(\mathbf{x})$ – corresponds to a linear map – here also denoted by $D^k\Phi(\mathbf{x})$ – on the tensor product space.

In the end, we are interested in small-time asymptotics, so we change variables

$$\mathbf{z} := \frac{1}{\sqrt{t}}(\mathbf{G} - \mathbf{G}^*),$$

so that we can express the above Taylor expansions as expansions in t,

$$\frac{1}{t}\Phi(\mathbf{G}) = \frac{1}{t}\Phi(\mathbf{G}^*) + \frac{1}{2}D^2\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 2} + \frac{1}{6}D^3\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 3}\sqrt{t} + \\ + \frac{1}{24}D^4\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 4}t + o(t),$$

and

$$\Psi(\mathbf{G}) = \Psi(\mathbf{G}^*) + D\Psi(\mathbf{G}^*)\mathbf{z}\sqrt{t} + \frac{1}{2}D^2\Psi(\mathbf{G}^*)\mathbf{z}^{\otimes 2}t + o(t).$$

Using the above Taylor expansions, the change of variables, and

$$e^{a\sqrt{t}+bt} = 1 + a\sqrt{t} + \left(\frac{a^2}{2} + b\right)t + o(t),$$

we obtain

$$\int_{\mathbf{G}_K} e^{-\frac{\Lambda(\mathbf{F}_0,\mathbf{F}_K(\mathbf{G}))}{t}}\Psi(\mathbf{G})d\mathbf{G} = t^{(n-1)/2}e^{-\Phi(\mathbf{G}^*)/t}\int_{(\mathbf{G}_K-\mathbf{G}^*)/\sqrt{t}} e^{-\frac{1}{2}D^2\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 2}}\times \\ \times\left[1 - \frac{1}{6}D^3\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 3}\sqrt{t} + \left(\frac{1}{2}\left\{-\frac{1}{6}D^3\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 3}\right\}^2 - \frac{1}{24}D^4\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 4}\right)t + o(t)\right]\times \\ \times\left[\Psi(\mathbf{G}^*) + D\Psi(\mathbf{G}^*)\mathbf{z}\sqrt{t} + \frac{1}{2}D^2\Psi(\mathbf{G}^*)\mathbf{z}^{\otimes 2}t + o(t)\right]d\mathbf{z}. \quad (3.9)$$

In the next step, we approximate the integral by replacing the domain of integration $(\mathbf{G}_K - \mathbf{G}^*)/\sqrt{t}$ by $\mathbb{R}^{n-1}$. Then we can see that the integration kernel in (3.9) is Gaussian with vanishing mean, so that the integral of any odd monomial with respect to the kernel vanishes. Thus, we obtain the expansion

$$\frac{|\mathbf{w}|}{|w_n|}\frac{1}{(2\pi t)^{n/2}}\int_{\mathbf{G}_K} e^{-\frac{\Lambda(\mathbf{F}_0,\mathbf{F}_K(\mathbf{G}))}{t}}\Psi(\mathbf{G})d\mathbf{G} \approx \left[h_0^\Psi + h_1^\Psi t + o(t)\right], \quad (3.10)$$

with h_0^Ψ defined in (3.8) and

$$h_1^\Psi := \frac{|\mathbf{w}|}{|w_n|}\frac{1}{(2\pi t)^{n/2}}\int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2}\mathbf{z}^T Q\mathbf{z}}\left[\frac{1}{2}D^2\Psi(\mathbf{G}^*)\mathbf{z}^{\otimes 2} - \frac{1}{6}D^3\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 3}\times D\Psi(\mathbf{G}^*)\mathbf{z} + \right. \\ \left. + \frac{1}{2}\left(\frac{1}{6}D^3\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 3}\right)^2\Psi(\mathbf{G}^*) - \frac{1}{24}D^4\Phi(\mathbf{G}^*)\mathbf{z}^{\otimes 4}\Psi(\mathbf{G}^*)\right]d\mathbf{z}. \quad (3.11)$$

Using Isserlis' Theorem (see [12]), the equation (3.11) for h_1^Ψ can be computed explicitly.

Lemma 3.1 (Isserlis' theorem for fourth and sixth moments)**.** *For a covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$ let $T^2(\Sigma) \in (\mathbb{R}^d)^{\otimes 4}$ and $T^3(\Sigma) \in (\mathbb{R}^d)^{\otimes 6}$ be the tensors defined by*

$$T^2(\Sigma)_{i_1,\dots,i_4} = \Sigma_{i_1i_2}\Sigma_{i_3i_4} + \Sigma_{i_1i_3}\Sigma_{i_2i_4} + \Sigma_{i_1i_4}\Sigma_{i_2i_3}$$

and

$$\begin{aligned}T^3(\Sigma)_{i_1,\dots,i_6} = {}& \Sigma_{i_1i_2}\Sigma_{i_3i_4}\Sigma_{i_5i_6} + \Sigma_{i_1i_2}\Sigma_{i_3i_5}\Sigma_{i_4i_6} + \Sigma_{i_1i_2}\Sigma_{i_3i_6}\Sigma_{i_4i_5} + \Sigma_{i_1i_3}\Sigma_{i_2i_4}\Sigma_{i_5i_6} + \\ &+ \Sigma_{i_1i_3}\Sigma_{i_2i_5}\Sigma_{i_4i_6} + \Sigma_{i_1i_3}\Sigma_{i_2i_6}\Sigma_{i_4i_5} + \Sigma_{i_1i_4}\Sigma_{i_2i_3}\Sigma_{i_5i_6} + \Sigma_{i_1i_4}\Sigma_{i_2i_5}\Sigma_{i_3i_6} + \\ &+ \Sigma_{i_1i_4}\Sigma_{i_2i_6}\Sigma_{i_3i_5} + \Sigma_{i_1i_5}\Sigma_{i_2i_3}\Sigma_{i_4i_6} + \Sigma_{i_1i_5}\Sigma_{i_2i_4}\Sigma_{i_3i_6} + \Sigma_{i_1i_5}\Sigma_{i_2i_6}\Sigma_{i_3i_5} + \\ &+ \Sigma_{i_1i_6}\Sigma_{i_2i_3}\Sigma_{i_4i_5} + \Sigma_{i_1i_6}\Sigma_{i_2i_4}\Sigma_{i_3i_5} + \Sigma_{i_1i_6}\Sigma_{i_2i_5}\Sigma_{i_3i_4},\end{aligned}$$

$1 \le i_1, \dots, i_6 \le d$. *For* $Z \sim \mathcal{N}(0,\Sigma)$ *we have*

$$E\left[Z^{\otimes 4}\right] = T^2(\Sigma), \quad E\left[Z^{\otimes 6}\right] = T^3(\Sigma).$$

Hence, we can get an explicit formula also for h_1^Ψ in terms of derivatives of Ψ and Φ – which are easy to compute, but lead to quite long formulas that are not included here.

Lemma 3.2. *With the short-hand notation* $\partial_{i_1,\dots,i_k} := \frac{\partial^k}{\partial F_{i_1}\cdots\partial F_{i_k}}$, *we have*

$$\begin{aligned}h_1^\Psi = \frac{|\mathbf{w}|}{|w_n|}\frac{1}{\sqrt{2\pi t\det Q}}\Bigg[&\frac{1}{2}D^2\Psi(\mathbf{G}^*)Q^{-1} - \frac{1}{6}\sum_{i_1,\dots,i_4}(\partial_{i_1,i_2,i_3}\Phi)(\mathbf{G}^*)(\partial_{i_4}\Psi)(\mathbf{G}^*)T^2(Q^{-1})_{i_1,\dots,i_4} \\ &+ \frac{1}{72}\Psi(\mathbf{G}^*)\sum_{i_1,\dots,i_6}(\partial_{i_1,i_2,i_3}\Phi)(\mathbf{G}^*)(\partial_{i_4,i_5,i_6}\Phi)(\mathbf{G}^*)T^3(Q^{-1})_{i_1,\dots,i_6} \\ &- \frac{1}{24}\Psi(\mathbf{G}^*)\sum_{i_1,\dots,i_4}(\partial_{i_1,\dots,i_4}\Phi)(\mathbf{G}^*)T^2(Q^{-1})_{i_1,\dots,i_4}\Bigg].\end{aligned}$$

These results are summarized in

Proposition 3.3. *We have the Laplace expansion*

$$\frac{1}{(2\pi t)^{n/2}}\int_{\mathcal{E}_K}\Psi(\mathbf{F})\exp\left(-\frac{d(\mathbf{F}_0,\mathbf{F})^2}{2t}\right)H_{n-1}(d\mathbf{F}) = h_0^\Psi + th_1^\Psi + o(t)$$

with h_0^Ψ *given in* (3.8) *and* h_1^Ψ *given in Lemma 3.2.*

The last ingredient needed for the asymptotic expansions of both implied and local volatilities are the heat kernel coefficients u_0 and u_1. As we are assuming the options to be ATM, we only need the heat kernel coefficients on the diagonal.

Lemma 3.4. *For a local volatility model, we have the following formulas for the heat kernel coefficients on the diagonal:*

$$\begin{aligned}u_0(\mathbf{F},\mathbf{F}) &= 1, \\ u_1(\mathbf{F},\mathbf{F}) &= \frac{1}{4}\sum_{i=1}^n\sigma_i(F_i)\sigma_i''(F_i) - \frac{1}{8}\sum_{i,j=1}^n\sigma_i'(F_i)\rho^{ij}\sigma_j'(F_j),\end{aligned}$$

where, as usual, ρ^{ij} *denotes the* (i,j)*-component of* ρ^{-1}.

Proof. Note that the infinitesimal generator A of the process $\mathbf{F}(t)$ can be expressed (using the summation convention) as

$$A = \frac{1}{2}\Delta - \frac{1}{2} f_i(\mathbf{F}) \frac{\partial}{\partial F_i},$$

where

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial F_i} g^{ij} \sqrt{\det g} \frac{\partial}{\partial F_j}$$

denotes the Laplace-Beltrami operator associated to g and the vector field f is given by

$$f_i(\mathbf{F}) = \sigma_i(\mathbf{F})\sigma_i'(F_i), \quad i = 1, \dots, n.$$

As indicated in (3.1), the transition density of the process $\mathbf{F}(t)$ satisfies (under certain assumptions)

$$p(\mathbf{F}_0, \mathbf{F}, T) = \frac{1}{(2\pi T)^{n/2}} \sqrt{\det g(\mathbf{F})} e^{-\frac{d(\mathbf{F}_0,\mathbf{F})^2}{2T}} (u_0(\mathbf{F}_0, \mathbf{F}) + T u_1(\mathbf{F}_0, \mathbf{F})) + o(T),$$

where $d(\mathbf{F}_0, \mathbf{F})$ is the geodesic distance between $\mathbf{F}_0$ and $\mathbf{F}$ and u_0 and u_1 are the heat kernel coefficients.

The order zero heat kernel coefficient is given by $u_0(\mathbf{F}_0, \mathbf{F}) = \sqrt{\Delta(\mathbf{F}_0, \mathbf{F})} e^{-\frac{1}{2}\int_\gamma \langle f, \dot\gamma_g \rangle}$, where $\int_\gamma \left\langle f, \dot\gamma_g \right\rangle$ is understood as integral along the geodesic γ joining $\mathbf{F}_0$ and $\mathbf{F}$ and $\Delta(\mathbf{F}_0, \mathbf{F})$ is the Van Vleck - De Witt determinant,

$$\Delta(\mathbf{F}_0, \mathbf{F}) = \frac{1}{\sqrt{\det g(\mathbf{F}_0) \det g(\mathbf{F})}} \det\left(-\frac{1}{2} \frac{\partial^2 d^2(\mathbf{F}_0, \mathbf{F})}{\partial \mathbf{F}_0 \partial \mathbf{F}} \right).$$

On the diagonal, we clearly have $\int_\gamma \left\langle f, \dot\gamma_g \right\rangle = 0$ and for any local volatility model we have $\Delta(\mathbf{F}_0, \mathbf{F}) \equiv 1$, as the geometry is isomorphic to the Euclidean geometry by the coordinate transformation $\mathbf{F} \mapsto L\mathbf{y}$, where $L\rho L^T = \mathrm{Id}$ and $y_i := \int_0^{F_i} \sigma_i(u)^{-1} du$. Hence, $u(\mathbf{F}, \mathbf{F}) = 1$.

For the first order heat kernel coefficient, we refer to [14, Eq. (4.1)], where it is shown that

$$u_1(\mathbf{F}, \mathbf{F}) = \frac{1}{6}\kappa + \frac{1}{4} \operatorname{div}_g f(\mathbf{F}) - \frac{1}{8} |f(\mathbf{F})|_g^2 .$$

Here, κ denotes the scalar curvature, which vanishes for local volatility model due to the isomorphism with the Euclidean geometry already used above. (Note that [14] consider the heat kernel corresponding to $\Delta + f$, whereas we consider the operator $\frac{1}{2}\Delta + \frac{1}{2} f$. Hence, we evaluate the formula obtained in [14, Eq. (4.1)] at $t/2$ instead of t.) For the remaining terms we have

$$\operatorname{div}_g f(\mathbf{F}) = \frac{1}{\sqrt{\det g(\mathbf{F})}} \frac{\partial}{\partial F_i} \left[f_i(\mathbf{F} \sqrt{\det g(\mathbf{F})} \right] = \sigma_i(F_i)\sigma_i''(F_i),$$
$$|f(\mathbf{F})|_g^2 = g_{ij}(\mathbf{F}) f_i(\mathbf{F}) f_j(\mathbf{F}) = \sigma_i'(F_i)\rho^{ij}\sigma_j'(F_j). \qquad \square$$

Finally, we can explicitly compute the determinant of the Hessian Q of Φ at $\mathbf{G}^* = (F_{0,1}, \dots, F_{0,n-1})$ in the ATM regime.

Lemma 3.5. *The Hessian Q of Φ satisfies*

$$\det Q = \frac{\sum_{i,j=1}^n w_i\sigma_i(F_{0,i})\rho_{ij}w_j\sigma_j(F_{0,j})}{w_n^2 \det\rho \prod_{i=1}^n \sigma_i(F_{0,i})^2} = \sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0)\det g(\mathbf{F}_0)/w_n^2.$$

The proof of Lemma 3.5 is deferred to the Appendix.

4. Basket local volatility

The numerator in the right hand side of the formula in Proposition 2.2 is given by

$$\frac{1}{(2\pi t)^{n/2}}\int_{\mathcal{E}_K}(\bar{u}_0(\mathbf{F}_0,\mathbf{F}) + t\bar{u}_1(\mathbf{F}_0,\mathbf{F}))\exp\left(-\frac{d(\mathbf{F}_0,\mathbf{F})^2}{2t}\right)H_{n-1}(d\mathbf{F}) = h_0^{\bar{u}_0} + t\left(h_1^{\bar{u}_0} + h_0^{\bar{u}_0}\right) + o(t),$$

where, by abuse of notation, we denote the function $\mathbf{F}\mapsto\bar{u}_i$ by $\bar{u}_i$ again, $i = 0, 1$. For the denominator, we get

$$\frac{1}{(2\pi t)^{n/2}}\int_{\mathcal{E}_K}(\hat{u}_0(\mathbf{F}_0,\mathbf{F}) + t\hat{u}_1(\mathbf{F}_0,\mathbf{F}))\exp\left(-\frac{d(\mathbf{F}_0,\mathbf{F})^2}{2t}\right)H_{n-1}(d\mathbf{F}) = h_0^{\hat{u}_0} + t\left(h_1^{\hat{u}_0} + h_0^{\hat{u}_1}\right) + o(t).$$

As

$$\frac{a_1 + b_1 t + o(t)}{a_2 + b_2 t + o(t)} = \frac{a_1}{a_2} + \frac{a_2b_1 - a_1b_2}{a_2^2}t + o(t),$$

we arrive at

$$\sigma_{\mathrm{loc}}(K,T)^2K^2 = \frac{h_0^{\bar{u}_0}}{h_0^{\hat{u}_0}} + \frac{h_0^{\hat{u}_0}(h_1^{\bar{u}_0} + h_0^{\bar{u}_1}) - h_0^{\bar{u}_0}(h_1^{\hat{u}_0} + h_0^{\hat{u}_1})}{\left(h_0^{\hat{u}_0}\right)^2}T + o(T).$$

As $\bar{u}_0 = \sigma^2_{\mathcal{N},\mathcal{B}}\hat{u}_0$, we can easily simplify

$$\frac{h_0^{\bar{u}_0}}{h_0^{\hat{u}_0}} = \sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0).$$

For the first order term, we note that all the terms $h_i^{\bar{u}_j}$ and $h_i^{\hat{u}_j}$ have the common factor $\frac{|w|}{|w_n|}\frac{1}{\sqrt{2\pi T\det Q}}$, which, hence, cancels out in the first order term – in particular, implying that the "first order term" is really first order in T. Thus, we get

Proposition 4.1. *For $K = \overline{F}_0 = \sum_{i=1}^n w_iF_{0,i}$, the basket local volatility has the asymptotic expansion $\sigma^2_{loc}(T,K) = \sigma^2_{loc,0}(K) + \sigma^2_{loc,1}(K)T + o(T)$, with*

$$\sigma^2_{loc,0}(K) = \frac{\sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0)}{K^2},$$

$$\sigma^2_{loc,1}(K) = \frac{h_0^{\hat{u}_0}(h_1^{\bar{u}_0} + h_0^{\bar{u}_1}) - h_0^{\bar{u}_0}(h_1^{\hat{u}_0} + h_0^{\hat{u}_1})}{\left(h_0^{\hat{u}_0}\right)^2K^2}.$$

We recall the definition

$$\sigma_{\mathcal{N},\mathcal{B}}(\mathbf{F})^2 = \sum_{i,j=1}^{n} w_i \sigma_i(F_i)\rho_{ij} w_j \sigma_j(F_j).$$

5. Implied volatility

The strategy for obtaining an asymptotic expansion for the implied volatility is as follows: we first compute an asymptotic expansion of the basket option price in our local volatility model, then we compare coefficients with the short time expansion of the corresponding call option price in the Black-Scholes or Bachelier model, respectively. Hence, we first apply our general asymptotic expansion obtained in Proposition 3.3 to the Carr-Jarrow formula from Proposition 2.1, getting (for $K = \overline{F}_0$)

Now we can insert these results back into Proposition 2.1, and we obtain

$$\begin{aligned} C_{\mathcal{B}}(\mathbf{F}_0, K, T) &= \frac{1}{2|w|}\int_0^T \left(h_0^{\bar{u}_0} + t\left(h_1^{\bar{u}_0} + h_0^{\bar{u}_1}\right) + o(\sqrt{t})\right) dt \\ &= \frac{1}{2}\int_0^T \left(\frac{g_0^{\bar{u}_0}}{\sqrt{t}} + \sqrt{t}\left(g_1^{\bar{u}_0} + g_0^{\bar{u}_1}\right) + o(\sqrt{t})\right) dt \\ &= g_0^{\bar{u}_0}\sqrt{T} + \frac{1}{3}\left(g_1^{\bar{u}_0} + g_0^{\bar{u}_1}\right) T^{3/2} + o\left(T^{3/2}\right), \end{aligned}$$

where

$$g_i^{\bar{u}_j} := \frac{\sqrt{t}}{|w|} h_i^{\bar{u}_j}, \quad i, j = 0, 1 \tag{5.1}$$

is independent of t. Finally, using (3.8) together with (3.3), and Lemma 3.5, we get

$$g_0^{\bar{u}_0} = \frac{\sigma^2_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0)\sqrt{\det g(\mathbf{F}_0)}}{|w_n|\sqrt{2\pi \det Q}} = \frac{\sigma_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0)}{\sqrt{2\pi}}.$$

Proposition 5.1. *The expansion of the call prices (at-the-money) in drift-less local volatility models is asymptotically equivalent, to first order, to*

$$C_{\mathcal{B}}(\mathbf{F}_0, K, T) = \frac{\sigma_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0)}{\sqrt{2\pi}} + \frac{1}{3}\left(g_1^{\bar{u}_0} + g_0^{\bar{u}_1}\right) T^{3/2} + o(T^{3/2})$$

as $T \to 0$.

In the final step, we compute an expansion of the implied volatility with respect to either Black-Scholes or Bachelier model. Let us consider the prices of call options with stock price $\overline{F}_0 = \sum_{i=1}^n w_i F_{0,i} = K$ in the Black-Scholes and Bachelier models, assuming that the respective volas are of the form $\sigma_{BS} = \sigma_{BS,0} + T\sigma_{BS,1}$ and $\sigma_{Bach} = \sigma_{Bach,0} + T\sigma_{Bach,1}$. We obtain the well known formulas

$$C_{BS}(\overline{F}_0, K, T) = C_{BS}(K, K, T) = \frac{K}{\sqrt{2\pi}}\sigma_{BS,0}\sqrt{T} + \frac{K}{\sqrt{2\pi}}\left[\sigma_{BS,1} - \frac{1}{24}\sigma^3_{BS,0}\right] + o(T^{3/2}),$$

$$C_{Bach}(\overline{F}_0, K, T) = C_{Bach}(K, K, T) = \frac{K}{\sqrt{2\pi}}\sigma_{Bach,0}\sqrt{T} + \frac{K}{\sqrt{2\pi}}\sigma_{Bach,1}T^{3/2} + o(T^{3/2}).$$

5.1. **Zeroth order implied vola.** Despite being well-known, we recall the zeroth order implied volatility coefficients and some of their properties. By comparison of coefficients, we find that

$$\sigma_{BS,0} = \sigma_{Bach,0} = \frac{1}{|w_n|K}\bar{u}_0(\mathbf{F}_0, \mathbf{F}_0)\,(\det Q)^{-\frac{1}{2}} = \frac{\sigma_{\mathcal{N},\mathcal{B}}(\mathbf{F}_0)}{\overline{F}_0}, \tag{5.2}$$

where we also used $\overline{F}_0 = K$. Note, in particular, that the basket implied volatility (5.2) can be interpreted as a weighted mean of the individual components' (ATM) implied volatilities in the sense that $(\sigma_{BS,0})^2 = \sum_{i,j=1}^n \rho_{ij} w_i \frac{F_{0,i}}{K}\sigma^i_{BS,0} w_j \frac{F_{0,j}}{K}\sigma^j_{BS,0}$.

Remark 5.2. The right hand side in equation (5.2) is nothing but the local volatility of the basket $\sum_{i=1}^n w_i F_i$ at $\mathbf{F}_0$ in the Black-Scholes (i.e., log-normal) sense. Hence, we have obtained that the zero order term in the small time expansion of the implied volatility of the basket is equal to its local volatility when we consider an ATM option. That result is not surprising in light of [9], where similar results were obtained (in one-dimensional models). In this sense, one could even take (5.2) as an ex-post justification of Lemma 3.5.

5.2. **First order implied vola.** The first order implied volatilities in the Black Scholes and the Bachelier model do not coincide any more. Indeed, we immediately have the first order correction term in the Bachelier model

$$\sigma_{Bach,1} = \frac{\sqrt{2\pi}}{3K}\left(g_1^{\bar{u}_0} + g_0^{\bar{u}_1}\right). \tag{5.3}$$

On the other hand, for the Black-Scholes model we have

$$\sigma_{BS,1} = \frac{\sqrt{2\pi}}{3K}\left(g_1^{\bar{u}_0} + g_0^{\bar{u}_1}\right) + \frac{\sigma^3_{BS,0}}{24} = \sigma_{Bach,1} + \frac{\sigma^3_{BS,0}}{24}, \tag{5.4}$$

implying that implied vola quoted in the Black-Scholes framework is strictly larger than the implied vola in the Bachelier framework up to first order – the prices are, of course, equal up to first order.

6. Numerical results

6.1. **The CEV model.** As in [4], we consider the CEV model for the numerical examples. The CEV model is a special case of the general local volatility model considered so far, where the local volatilities are given by

$$\sigma_i(F_i) = \xi_i F_i^{\beta_i}, \quad i = 1, \dots, n,$$

for some parameters $\xi_i \geq 0$ and $\beta_i > 0$. In fact, the most realistic scenario here is $0 < \beta_i \leq 1$. Note that we allow $\beta_i < 1/2$.

6.2. **Implementation of the approximate formulas and simulation.** Implementation of the zero order terms of the implied volatilities in either Black-Scholes or Bachelier setting is, of course, easy using (5.2). On the other hand, the formulas for $\sigma_{BS,1}$ and $\sigma_{Bach,1}$ are much less straightforward to implement. While the formulas in the ATM case are fully explicit (unlike in [4]) an efficient implementation is much less trivial. The formula for h_1 in Lemma 3.2, for instance, depends on the derivatives up to order four of the squared Riemannian distance at $\mathbf{F}_0$ and on the Jacobi matrix of $\mathbf{F} \mapsto u_0(\mathbf{F}_0, \mathbf{F})$. Already the evaluation of the $(n-1) \times (n-1) \times (n-1) \times (n-1)$ tensor $D^4\Phi$ can be very time-consuming, if a naive implementation is used, which does not take into account that most derivatives actually vanish. But even when more efficient implementations are used, the sheer size of the tensor may impose limitations on the dimension of the problem. So far, we have implemented (3.11) in Mathematica using symbolic differentiation of the squared Riemannian distance and the zeroth order heat kernel coefficient u_0, which works for small dimensions, up to $n = 5$, say.

As in the paper [4], we compare the approximate prices against prices obtained from sophisticated Monte Carlo simulation. Here, the CEV-SDE is discretized using the Ninomiya-Victoir scheme [16], which is a second order weak approximation scheme based on a splitting of the generator. Strictly speaking, the CEV process violates the strong regularity assumption of that scheme, especially at the boundary of the domain, but, as often in equity modelling, we do empirically observe second order convergence for CEV-baskets, yet another beneficial effect of "not feeling the boundary". For variance reduction, we combine the discretization with the mean value Monte Carlo method, see [17]. This is a variant of the control variate technique, where a linear combination of one-dimensional geometrical Brownian motions is used as control variate. More precisely, we freeze each component but one of the basket, and replace the dynamics of the remaining basket by a corresponding Black-Scholes dynamics. In the resulting model, the true option price can be explicitly calculated. Finally, we choose a linear combination of those partially frozen model so as to minimize the variance of the Monte Carlo estimator.

The expectation of the random variable obtained by combining the Ninomiya-Victoir discretization of the CEV process and the mean value Monte Carlo method is the approximated using Sobol numbers. In some sense, this contradicts the above motivation for the variance reduction, but we do find empirically that the integration error for a Quasi Monte Carlo estimator is also reduced by the variance reduction, i.e., the variance reduction also seems to reduce the number of most relevant dimensions of the integration problem. Finally, we sacrifice some of the accuracy available by the combination of the three techniques mentioned so far by introducing a random shift of the Sobol numbers, i.e., we use the Randomized Quasi Monte Carlo technique, see L'Ecuyer [13]. In this way, we can obtain reliable computable error bounds for the integration error.

6.3. **Numerical examples.**

Example 6.1. In the first example, we consider a three-dimensional spread option, which is determined by the following parameters:

$$\mathbf{F}_0 = \begin{pmatrix} 8 \\ 17 \\ 12 \end{pmatrix}, \quad \boldsymbol{\sigma} = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.7 \end{pmatrix}, \quad \boldsymbol{\beta} = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

with a correlation matrix

$$\boldsymbol{\rho} = \begin{pmatrix} 1 & 0.9167390 & 0.7425194 \\ 0.9167390 & 1 & 0.8099573 \\ 0.7425194 & 0.8099573 & 1 \end{pmatrix}.$$

We compute the ATM price, i.e., the option price at $K = 21$, for maturities

Time	Price	0th order price	1st order price	Error bound
0.5	0.88073	0.88092	0.88072	2.43e-05
1	1.24525	1.24581	1.24524	4.63e-05
2	1.76023	1.76184	1.76024	8.90e-05
5	2.77895	2.78571	2.77941	3.21e-04
10	3.91968	3.93959	3.92176	5.92e-04

TABLE 1. Prices and implied volas in Example 6.1. Error bounds given correspond to the (quasi) Monte Carlo error in the numerical scheme. The discretization error is of higher order.

Time	0th order rel. error	1st order rel. error	Error bound
0.5	2.19e-04	6.85e-06	2.43e-05
1	4.49e-04	3.80e-06	4.63e-05
2	9.15e-04	9.02e-06	8.90e-05
5	2.43e-03	1.65e-04	3.21e-04
10	5.08e-03	5.33e-04	5.92e-04

TABLE 2. Relative errors in Example 6.1. Error bounds given correspond to the (quasi) Monte Carlo error in the numerical scheme. The discretization error is of higher order.

$T \in \{0.5, 1, 2, 5, 10\}$ years, which we compare with the zeroth and first order prices in the corresponding Bachelier model. We also report $\sigma_{Bach,0} = 0.1487036$ and $\sigma_{Bach,1} = -6.72781 \times 10^{-5}$. Note that the "error bounds" reported in Tables 1 and 2 are upper estimates for the integration error (i.e., quasi Monte Carlo error) for the reference values. Hence, numbers obtained from the first order approximation formula are within the error bounds around the reference values.

In Figure 6.1, we plot (linear interpolations of) the relative errors of the zeroth and first order approximate pricing formulas close to the money (as obtained in [4]) and compare them to the ATM-formulas represented by circles. We see that

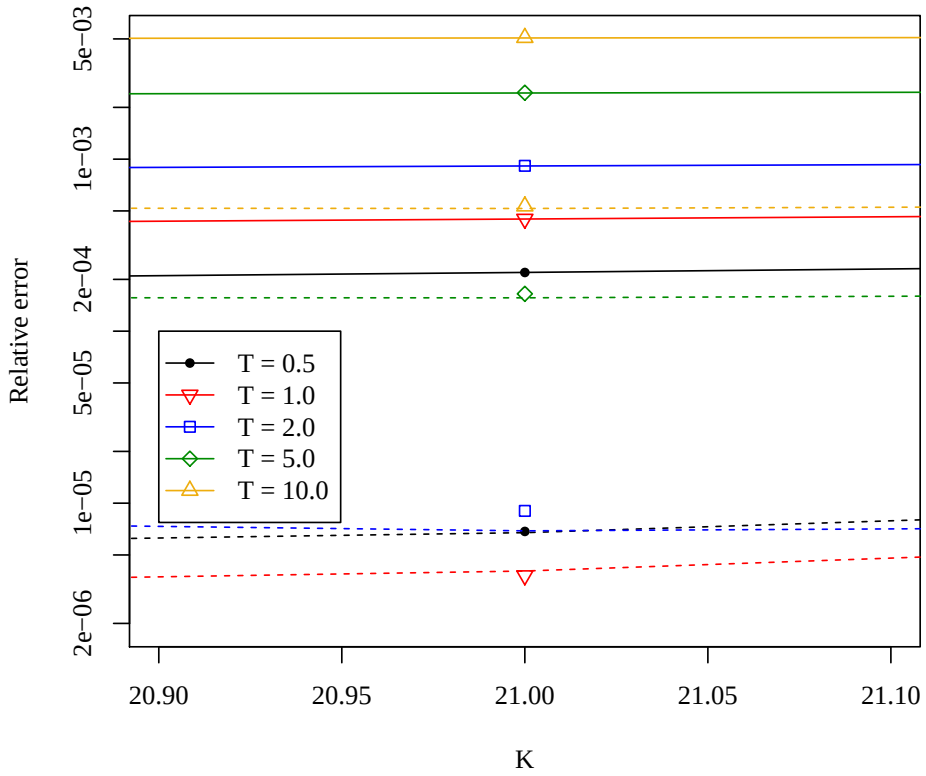

FIGURE 1. Relative errors in Example 6.1. Solid lines correspond to prices obtained from (non-ATM) zeroth order approximate formulas, dashed lines to (non-ATM) first order approximate formulas. The corresponding ATM-approximate prices are represented by circles and other symbols. Note that the option is ATM for $K = 21$.

the accuracy is extremely good in both cases, and that our approximation formulas for ATM CEV-basket options nicely interpolate the formulas available away from the money. Indeed, deviations from the non-ATM values only appears at very small orders of magnitude in the logarithmic scale of Figure 6.1 (where the Monte Carlo error contained in the reference values probably dominates). For the sake of completeness, Figure 2 reports the absolute errors of the respective asymptotic formulas over a wide range of strike prices, indicating that the asymptotic formulas exhibit their worst quality ATM.

APPENDIX A. PROOF OF LEMMA 3.5

We present a proof of Lemma 3.5. Recall that we want to compute the determinant of the Hessian Q of the map

$$\Phi(\mathbf{G}) := \frac{1}{2} d\left(\mathbf{F}_0, (\mathbf{G}, F_N(\mathbf{G}, K))\right)^2$$

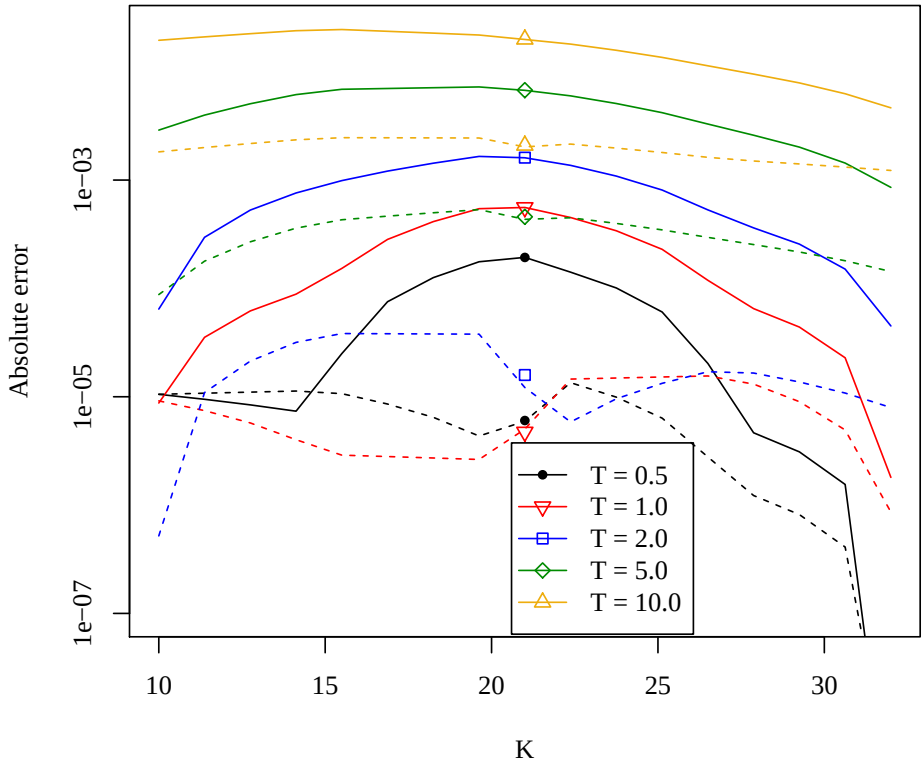

FIGURE 2. Absolute errors in Example 6.1. Solid lines correspond to prices obtained from (non-ATM) zeroth order approximate formulas, dashed lines to (non-ATM) first order approximate formulas. The corresponding ATM-approximate prices are represented by circles and other symbols. Note that the option is ATM for $K = 21$.

evaluated at $\mathbf{G} = (F_{0,1}, \dots, F_{0,n-1})$. Let $\Im_i(x)$ denote the anti-derivative of $1/\sigma_i$ satisfying (for simplicity) $\Im_i(F_{0,i}) = 0$. Now consider the change of variables $\mathbf{F} \to \mathbf{y}$ with $y_i := \Im_i(F_i)$, $i = 1, \dots, n$. As verified in [4], this transformation turns the Riemannian geometry introduced above into an (almost) Euclidean geometry, with

$$d(\mathbf{F}_0, \mathbf{F})^2 = \mathbf{y}^T \rho^{-1} \mathbf{y}.$$

Of course, the constraint on $\mathbf{F}$ translates into a constraint on $\mathbf{y}$, which can be removed by eliminating one variable. Indeed, setting $\mathbf{x} := (y_1, \dots, y_{n-1})$, we get

$$y_n(\mathbf{x}) = \Im_n(F_n) = \Im_n\left(\frac{1}{w_n}\left(K - \sum_{j=1}^{n-1} w_j \Im_j^{-1}(y_j)\right)\right).$$

This way, we understand $\Phi(\mathbf{G})$ as a function $\varphi(\mathbf{x})$ in the new (reduced) coordinates, and obtain for the Hessian

$$H_{\mathbf{G}}\Phi(\mathbf{G}) = J(\mathbf{G})^T H_{\mathbf{x}}\varphi(\mathbf{x})J(\mathbf{G}),$$

where $H_{\mathbf{G}}$ and $H_{\mathbf{x}}$ denote the Hessians in the $\mathbf{G}$- and $\mathbf{x}$-coordinates, respectively, and $J(\mathbf{G})$ denotes the Jacobian matrix of the change of coordinates $\mathbf{G} \to \mathbf{x}$. As $\Xi_i' = 1/\sigma_i$, we have $J(\mathbf{G}) = \operatorname{diag}(1/\sigma_1(F_1), \dots, 1/\sigma_{n-1}(F_{n-1}))$. Regarding the matrix $H_{\mathbf{x}}\varphi$, an elementary calculation using the fact that $\mathbf{F} = \mathbf{F}_0$ corresponds to $\mathbf{y} = 0$, we obtain

$$H_{\mathbf{x}}\varphi(0) = \left(\rho^{ij} - \rho^{in}\frac{w_j\sigma_j(F_{0,j})}{w_n\sigma_n(F_{0,n})} - \rho^{jn}\frac{w_i\sigma_i(F_{0,i})}{w_n\sigma_n(F_{0,n})} + \rho^{nn}\frac{w_i\sigma_i(F_{0,i})w_j\sigma_j(F_{0,j})}{w_n^2\sigma_n(F_{0,n})^2}\right)_{i,j=1}^{n-1}.$$

From the structure of the above expression and the expression in Lemma 3.5, we see that we may assume that $w_i = 1$, $i = 1, \dots, n$, and $\sigma_n(F_{0,n}) = 1$. In this case, we are left to prove that the determinant of the matrix

$$A := \left(\rho^{ij} - \rho^{in}s_j - \rho^{jn}s_i + \rho^{nn}s_is_j\right)_{i,j=1}^{n-1}$$

is equal to the expression $a := \mathbf{s}^T\rho\mathbf{s}/\det\rho$, where we used the short-hand notation $s_i = \sigma_i(F_{0,i})$, $i = 1, \dots, n-1$, and $s_n = 1$, and $\mathbf{s} = (s_1, \dots, s_n)$.

As both $\det A$ and a are polynomials in $s_1, \dots, s_{n-1}$, we prove this equality by establishing that they have the same coefficients. Here, Cramer's rule is the essential tool:

$$B^{-1} = \frac{1}{\det B}\operatorname{Adj}(B),$$

where the *adjugate* matrix $\operatorname{Adj} B$ is the transpose of the matrix of co-factors, i.e.,

$$(\operatorname{Adj} B)_{ij} = (-1)^{i+j}\det B_{\hat{j}\hat{i}},$$

with $B_{\hat{j}\hat{i}}$ being obtained from B by removing the j'th row and the i'th column. By symmetry, we hence have

$$\frac{\rho_{ij}}{\det\rho} = (-1)^{i+j}\det\rho^{-1}_{\hat{i}\hat{j}}, \quad \forall (i,j) \in \{1, \dots, n-1\}^2, \tag{A.1}$$

where $\rho^{-1}_{\hat{i}\hat{j}}$ is understood in the sense of $(\rho^{-1})_{\hat{i}\hat{j}}$.

Let us also establish a few notations. Let S_{n-1} be the set of all permutations of $\{1, \dots, n-1\}$ and let, similarly, be $S(A;B)$ denote the set of all bijective maps from $A \subset \mathbb{N}$ to $B \subset \mathbb{N}$, with A, B having the same (finite) size. Moreover, the definition of the signature sign is extended to $S(A;B)$ in the obvious way (as being ± 1 depending on the number of inversions being even or odd). Moreover, for a monomial x in the variables $s_1, \dots, s_{n-1}$ we denote by $\pi_x p$ the coefficient of any polynomial p w. r. t. the monomial x. In order to establish Lemma 3.5, we need to prove that

$$\forall x \in \bigcup_{k=0}^{2(n-1)} \{s_1, \dots, s_{n-1}\}^k : \ \pi_x \det A = \pi_x a.$$

We distinguish different cases according to the degree.

Case 0. For $\deg x = 0$, i.e., $x = 1$, we have

$$\pi_1 \det A = \sum_{\sigma\in S_{n-1}} \operatorname{sign}(\sigma) \prod_{i=1}^{n-1} \rho^{i\sigma(i)} = \det \rho^{-1}_{\hat{n}\hat{n}} = \operatorname{Adj}(\rho^{-1})_{nn} = \frac{\rho_{nn}}{\det \rho} = \pi_1 a.$$

Case 1. For some fixed s_k we have

$$\begin{aligned}\pi_{s_k} \det A &= \sum_{\sigma\in S_{n-1}} \operatorname{sign}(\sigma)(-1)\left[\rho^{\sigma^{-1}(k)n} \prod_{i\in\{1,\dots,n-1\}\setminus\{\sigma^{-1}(k)\}} \rho^{i\sigma(i)} + \rho^{\sigma(k)n} \prod_{i\in\{1,\dots,n-1\}\setminus\{k\}} \rho^{i\sigma(i)}\right] \\ &= -2 \sum_{\sigma\in S_{n-1}} \operatorname{sign}(\sigma)\rho^{\sigma(k)n} \prod_{i\in\{1,\dots,n-1\}\setminus\{k\}} \rho^{i\sigma(i)}\end{aligned}$$

by symmetry of ρ^{-1}. There is a one-to-one correspondence between S_{n-1} and $S(\{1,\dots,n\}\setminus\{k\};\{1,\dots,n-1\})$ given by $\sigma \mapsto \tilde{\sigma}$ defined by

$$\tilde{\sigma}(i) = \begin{cases}\sigma(i), & i \in \{1,\dots,n-1\}\setminus\{k\},\\ \sigma(k), & i = n.\end{cases}$$

Moreover, one can see that $\operatorname{sign}(\tilde{\sigma}) = (-1)^{k+n-1}\operatorname{sign}(\sigma)$. Hence, we obtain

$$\begin{aligned}\pi_{s_k} \det A &= -2 \sum_{\sigma\in S_{n-1}} \operatorname{sign}(\sigma)\rho^{n\tilde{\sigma}(n)} \prod_{i\in\{1,\dots,n-1\}\setminus\{k\}} \rho^{i\tilde{\sigma(i)}} \\ &= 2(-1)^{k+n} \sum_{\tilde{\sigma}\in S(\{1,\dots,n\}\setminus\{k\};\{1,\dots,n-1\})} \operatorname{sign}(\tilde{\sigma})\rho^{n\tilde{\sigma}(n)} \prod_{i\in\{1,\dots,n-1\}\setminus\{k\}} \rho^{i\tilde{\sigma(i)}} \\ &= 2(-1)^{k+n} \det \rho^{-1}_{\hat{k}\hat{n}} \\ &= 2\operatorname{Adj}(\rho^{-1})_{kn} = \frac{2\rho_{kn}}{\det\rho} = \pi_{s_k} a.\end{aligned}$$

Case 2. We consider $x = s_k s_l$. For simplicity, we assume $k = l$ ($k \neq l$ works analogously). We have

$$\begin{aligned}\pi_{s_k^2} \det A = \sum_{\sigma\in S_{n-1}} \operatorname{sign}(\sigma)\Big[&\mathbf{1}_{k=\sigma(k)}\rho^{nn} \prod_{i\in\{1,\dots,n-1\}\setminus\{k\}} \rho^{i\sigma(i)} + \\ &+ \mathbf{1}_{k\neq\sigma(k)}\rho^{\sigma(k)n}\rho^{\sigma^{-1}(k)n} \prod_{i\in\{1,\dots,n-1\}\setminus\{k,\sigma^{-1}(k)\}} \rho^{i\sigma(i)}\Big].\end{aligned}$$

We construct a bijective map from S_{n-1} to $S(\{1,\dots,n\}\setminus\{k\};\{1,\dots,n\}\setminus\{k\})$ by mapping $\sigma \in S_{n-1}$ to $\tilde{\sigma}$ defined by

$$\tilde{\sigma}(i) = \begin{cases}\sigma(i), & i \in \{1,\dots,n-1\}\setminus\{k\},\\ n, & i = n,\end{cases}$$

for the case $k = \sigma(k)$ and

$$\tilde{\sigma}(i) = \begin{cases}\sigma(i), & i \in \{1,\dots,n-1\}\setminus\{k,\sigma^{-1}(k)\},\\ n, & i = \sigma^{-1}(k),\\ \sigma(k), & i = n,\end{cases}$$

else. Note that it is easy to see that $\operatorname{sign}(\sigma) = \operatorname{sign}(\tilde{\sigma})$. Hence, we have

$$\begin{aligned}\pi_{s_k^2} \det A &= \sum_{\sigma \in S_{n-1}} \operatorname{sign}(\sigma) \prod_{i \in \{1,\dots,n\}\setminus\{k\}} \rho^{i\tilde{\sigma}(i)} \\ &= \sum_{\tilde{\sigma} \in S(\{1,\dots,n\}\setminus\{k\};\{1,\dots,n\}\setminus\{k\})} \operatorname{sign}(\tilde{\sigma}) \prod_{i \in \{1,\dots,n\}\setminus\{k\}} \rho^{i\tilde{\sigma}(i)} \\ &= \det \rho^{-1}_{\hat{k}\hat{k}} = \pi_{s_k^2} a.\end{aligned}$$

Higher order terms. Regarding the higher order terms, we note that $\pi_x a = 0$ for any monomial of degree larger than two. Therefore, the same should be true for $\det A$, where it does not to seem to follow from an obvious argument. Note that we only need to consider polynomials where each individual variable s_k appears at most two times, as any other monomial cannot appear in $\det A$ by the definition of A and of the determinant. But any coefficient of $\det A$ with respect to such monomials can be understood as the determinant of a matrix $\widetilde{\rho^{-1}}$, which is obtained from ρ^{-1} by omitting one row and one column *and* by replacing some rows/columns by copies of other rows/columns. Of course, any such matrix $\tilde{\rho}$ has vanishing determinant, implying that $\pi_x \det A = 0$. For concreteness, we indicate this mechanism by appealing to two special cases. First, take $x = s_k^2 s_l$, $l \neq k$. Similarly to the case of $x = s_k$, one can show that

$$\begin{aligned}\pi_{s_k^2 s_l} \det A = -2 \sum_{\sigma \in S_{n-1}} \operatorname{sign}(\sigma) \Bigg[&\mathbf{1}_{k=\sigma(k)} \rho^{nn} \rho^{\sigma^{-1}(l)n} \prod_{i \in \{1,\dots,n-1\}\setminus\{k,l\}} \rho^{i\sigma(i)} + \\ &+ \mathbf{1}_{k\neq\sigma(k)} \rho^{\sigma(k)n} \rho^{\sigma^{-1}(k)n} \rho^{\sigma^{-1}(l)n} \prod_{i \in \{1,\dots,n-1\}\setminus\{k,\sigma^{-1}(k)\sigma^{-1}(l)\}} \rho^{i\sigma(i)},\end{aligned}$$

which is (the multiple of) the determinant of $\widetilde{\rho^{-1}}$, which is obtained from $\rho^{-1}_{\hat{k}\hat{k}}$ by replacing the l'th row by the last row. As the last row appears twice in $\widetilde{\rho^{-1}}$, the determinant, and hence $\pi_{s_k^2 s_l} \det A$, vanishes.

The mechanism is even more transparent for the most extreme monomial $x = s_1^2 \cdots s_{n-1}^2$. In this case,

$$\pi_{s_1^2 \cdots s_{n-1}^2} \det A = \sum_{\sigma \in S_{n-1}} \operatorname{sign}(\sigma) (\rho^{nn})^{n-1} = 0,$$

as the determinant of the $(n-1)\times(n-1)$ matrix with all entries being equal to ρ^{nn}.

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