Option pricing in affine generalized Merton models

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Abstract

In this article we consider affine generalizations of the Merton jump diffusion model [7] and the respective pricing of European options. On the one hand, the Brownian motion part in the Merton model may be generalized to a log-Heston model, and on the other hand, the jump part may be generalized to an affine process with possibly state dependent jumps. While the characteristic function of the log-Heston component is known in closed form, the characteristic function of the second component may be unknown explicitly. For the latter component we propose an approximation procedure based on the method introduced in [1]. We conclude with some numerical examples.

1 Introduction

The Merton jump diffusion model [7] can be considered one of the first asset models beyond Black-Scholes that may produce non-flat implied volatility surfaces. On the other hand, European options within this model can be priced quasi-analytically by means of an infinite series of Black-Scholes type expressions. From a mathematical point of view, the logarithm of the Merton model is the sum of a compound Poisson process an an independent Brownian motion, and as such can be seen as the sum of two independent degenerate affine processes. The goal of this article is to enlarge the flexibility of the Merton model by generalizing the Brownian motion to a continuous affine Heston model and replacing the compound Poisson process by another, independent, affine model that may incorporate both stochastic volatility and jumps. In financial modeling affine processes have become very popular the last decades, both due to their flexibility and their analytical tractability. The theoretical analysis of affine processes is developed in the seminal papers [4] and [3]. Once the characteristic functions of the affine ingredients of our new generalized Merton model are known, we may price European options by the meanwhile standard Carr-Madan Fourier based method [2]. For a variety of affine models, such as the Heston model and several stochastic volatility models with state independent jumps, the characteristic function is explicitly known. However if, for instance, in an affine jump model the jump intensity depends on the present state, a closed form expression for the characteristic function is not known to the best of our knowledge. Yet, such models make sense in certain applications such as crisis modeling. For example, one may wish to model an increased intensity of downward jumps in regimes of increased volatility. In order to cope with such kind of processes numerically, we recap and apply the general series expansion representation for the characteristic function of an affine process developed in [1] and present some numerical examples.

2 Merton jump diffusion models

Merton [7] introduced and studied stock price models of the form

$$S_t = S_0 e^{rt+Y_t},$$

where Y is the sum of a Brownian motion with drift and an independent compound Poisson process,

$$Y_t = \gamma t + \sigma W_t + J_t, \tag{1}$$

and r is a constant, continuously compounded risk-free rate. In (1) J may be represented as

$$J_t = \sum_{l=1}^{N_t} U_l,$$

where $U_1, U_2, ...$ are i.i.d. real valued random variables and N_t denotes the number of time marks up to time t that arrive at exponential times with parameter λ, i.e.

$$N_t := \#\{i : s_i \leq t, \quad s_i - s_{i-1} \sim \exp_\lambda, \quad i = 1, 2, ...\}$$

with $s_0 := 0$, and where $\tau \sim \exp_\lambda$ denotes an exponentially distributed random variable with

$$\mathbb{P}[\tau \geq s] = e^{-\lambda s} \text{ for all } s \geq 0.$$

From basic probability theory we know that N_t is Poisson distributed according to

$$\mathbb{P}[N_t = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!},$$

and that the characteristic function of Y_t is given by,

$$\begin{aligned}\Phi_t(z) &= \mathbb{E}\left[e^{\mathrm{i}zY_t}\right] = e^{\mathrm{i}z\gamma t}\mathbb{E}\left[e^{\mathrm{i}z\sigma W_t}\right]\mathbb{E}\left[e^{\mathrm{i}zJ_t}\right] \\ &= \exp\left[\mathrm{i}z\gamma t - \frac{z^2\sigma^2}{2}t + \lambda t \int \left(e^{\mathrm{i}zu} - 1\right) p(du)\right],\end{aligned} \tag{2}$$

for a certain jump probability measure p on $\mathcal{B}(\mathbb{R})$ due to the distribution of U_1.

We henceforth assume a risk-neutral pricing measure and due to no-arbitrage arguments we must have that $S_t e^{-rt}$ is a martingale under this measure. This implies that

$$S_0 = \mathbb{E}\left[S_t e^{-rt}\right] = S_0 \mathbb{E}\left[\exp\left(Y_t\right)\right] = S_0 \Phi_t(-\mathrm{i}), \quad \text{hence} \quad \Phi_t(-\mathrm{i}) = 1. \tag{3}$$

By (2) we then get

$$\gamma = -\frac{\sigma^2}{2} - \lambda \int (e^u - 1)\, p(du). \tag{4}$$

As an example, with $\lambda = 0$ (no jumps), $\gamma = -\frac{\sigma^2}{2}$ and we retrieve the risk neutral Black-Scholes model. Merton particularly studied the case where U is normally distributed and derived a representation for a call (or put) option in terms of an infinite series of Black-Scholes expressions. In this paper we are interested in generalizations of (1) of the form

$$Y_t = \gamma t + \sigma W_t + X_t^1, \tag{5}$$

or even,

$$Y_t = \gamma t + H_t + X_t^1, \tag{6}$$

where H is the first component of a log-Heston type model with $H_0 = 0$, whereas X_t^1 is the first component of some generally multidimensional affine (eventually jump) process X, independent of W and H respectively, with $X_0^1 = 0$. In particular, the characteristic function of X^1 is possibly not known in closed form.

3 Recap of affine processes and approximate characteristic functions

We consider an affine process X in the state space $\mathfrak{X} \subset \mathbb{R}^d$, $d \in \mathbb{N}_+$, with generator given by

$$\begin{aligned} Af(x) = {} & \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x)\frac{\partial f}{\partial x^i} \\ & + \int_{\mathbb{R}^d}\left[f(x+z) - f(x) - z^\top \frac{\partial f}{\partial x}\right] v(x, dz), \end{aligned} \tag{7}$$

where a^{ij} and b^i are suitably defined *affine* functions in x on $\mathbb{R}^d$, and

$$v(x, dz) =: v^0(dz) + x^\top v^1(dz)$$

with v^0 and v_i^1, $i = 1, ..., d$, being suitably defined locally finite measures on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$. Alternatively, the dynamics of X are described by the Itô-Lévy SDE:

$$dX_t = b(X_t)dt + \sigma(X_t)dW(t) + \int_{\mathbb{R}^d} z\widetilde{N}(X_{t-}, dt, dz), \quad X_0 = x, \tag{8}$$

where W is a Wiener process in $\mathbb{R}^m$ and the function $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^m$ satisfies

$$\sum_{k=1}^{m} \sigma_{ik}(x)\sigma_{jk}(x) = a^{ij}(x).$$

Further, in (8)

$$\widetilde{N}(x, dt, dz) := \widetilde{N}(x, dt, dz, \omega) := N(x, dt, dz, \omega) - v(x, dz)dt,$$

is a compensated Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d$, such that

$$\mathbb{P}\left[N(x, (0, t], B) = k\right] = \exp(-tv(x, B))\frac{t^k v^k(x, B)}{k!}, \quad k = 0, 1, 2, ...$$

for bounded $B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. It is assumed that the coefficients in (8) (and so in (7)) satisfy sufficient conditions such that (8) has a unique strong solution X, and that X is an affine process with generator (7). For details regarding these assumptions, in particular the admissibility conditions that are to be fulfilled, we refer to [1], [3], see also [4].

The characteristic function of $X_t^{0;x}$, with $X_0^{0;x} = x \in \mathbb{R}^d$, is denoted by,

$$\widehat{p}(t, x, u) := \mathbb{E}\left[e^{\mathrm{i}u^\top X_t^{0;x}}\right], \quad x \in \mathfrak{X},\ u \in \mathbb{R}^d, \quad t \geq 0. \tag{9}$$

For a variety of affine processes the characteristic function is explicitly known. However, in general the characteristic function of an affine process involves the solution of a multi-dimensional Riccati equation that may not be solved explicitly. In particular, for affine jump processes with state dependent jump part a closed form expression for the characteristic function generally doesn't exist. In this section we recall the approach by Belomestny, Kampen, and Schoenmakers [1], who developed in general a series expansion for the log-characteristic function in terms of the ingredients of the generator of the affine process under consideration. By truncating this expansion one may obtain an approximation of the characteristic function that may subsequently be used for approximate option pricing.

Henceforth, $x \in \mathfrak{X}$ is fixed. It is assumed that the characteristic function (9) satisfies:

Assumption **HE:** There exists a non-increasing function $R : (0, \infty) \ni r \to R(r) \in (0, \infty]$, such that for any $u \in \mathbb{R}^d$, the function $[0, \infty) \ni s \to \widehat{p}(s, x, u) \in \mathbb{C}$ has a holomorphic extension to the region

$$G_u := \{z \in \mathbb{C} : |z| < R(\|u\|)\} \cup \{z \in \mathbb{C} : \operatorname{Re} z \geq 0 \quad \text{and} \quad |\operatorname{Im} z| < R(\|u\|)\}$$

(cf. Prop. 3.7, 3.8, and Th. 4.1 and Corr. 4.2-4.4 in [1]).

Under Assumption **HE**, Th. 3.4 in [1] is particularly fulfilled for each u. Moreover, by taking in [1], Th. 3.4-(ii),

$$\eta_u = \eta(\|u\|) := \frac{\pi}{2R(\|u\|)}, \tag{10}$$

we arrive at the log-series representation [1]-(5.12) for the characteristic function,

$$\ln \widehat{p}(t,x,u) = \ln \left(\sum_{r\geq 0} h_{r,0}(u;\eta_u)(1-e^{-\eta_u t})^r \right) + \mathfrak{i}u^\top x \tag{11}$$
$$+ x^\top \frac{\sum_{r\geq 1} h_r(u;\eta_u)\,(1-e^{-\eta_u t})^r}{\sum_{r\geq 0} h_{r,0}(u;\eta_u)(1-e^{-\eta_u t})^r}, \quad u\in\mathbb{R}^d, \quad t\geq 0,$$

where the coefficients $h_{r,0}(u;\eta_u)\in\mathbb{C}$ and $h_r(u;\eta_u) = [h_{r,e_1}(u;\eta_u), ..., h_{r,e_d}(u;\eta_u)] \in \mathbb{C}^d$ with $e_i := (\delta_{ij})_{j=1,...,d}$, can be computed algebraically from the coefficients of the affine generator A in a way that is described below.

Alternatively, in [1] a ground expansion of the form

$$\widehat{p}(t,x,u) = e^{\mathfrak{i}u^\top x} \sum_{r=0}^{\infty} q_r(x,u;\eta_u)(1-e^{-\eta_u t})^r, \quad u\in\mathbb{R}^d, \quad t\geq 0, \tag{12}$$

is derived with

$$q_r(x,u;\eta_u) = \sum_{|\gamma|\leq r} h_{r,\gamma}(u;\eta_u) x^\gamma,$$

and the $h_{r,\gamma}$ are computed by the recursion (15) as described below.

Remark 1 *Because of Assumption* ***HE,*** *if Th. 3.4-(i) applies for some* u, *it applies for any* u' *with* $\|u'\| \leq \|u\|$ *. As a consequence, one may take in (11) any* $\eta_u = \eta(\|u''\|)$ *with* $\|u''\| \geq \|u\|$ *.*

In order to outline the construction of the expansion (11), let us denote

$$f_u(x) := e^{\mathfrak{i}u^\top x}, \quad z\in\mathbb{R}^d. \tag{13}$$

Then for each multi-index $\beta\in\mathbb{N}_0^d$ we may compute algebraically

$$\mathfrak{b}_\beta(x,u) := \mathfrak{i}^{-|\beta|}\partial_{u^\beta} \frac{Af_u(x)}{f_u(x)} =: \mathfrak{b}^0_\beta(u) + \sum_{\kappa,\,|\kappa|=1} \mathfrak{b}^1_{\beta,\kappa}(u)\, x^\kappa \tag{14}$$

(in multi-index notation), provided that for the jump part in the generator (7),

$$\frac{1}{f_u(x)} \int_{\mathbb{R}^d} \left(f_u(x+z) - f_u(x) - z^\top \frac{\partial f_u}{\partial x} \right) v(x,dz)$$
$$= \int_{\mathbb{R}^d} \left(e^{\mathfrak{i}u^\top z} - 1 - \mathfrak{i}u^\top z \right) v^0(dz) + x^\top \int_{\mathbb{R}^d} \left(e^{\mathfrak{i}u^\top z} - 1 - \mathfrak{i}u^\top z \right) v^1(dz)$$

is explicitly known. That is, the cumulant generating functions of v^0 and v^1_i, $i = 1, ..., d$, are explicitly known. We note that the expression $Af_u(x)/f_u(x)$ in (14) is termed the *symbol* of the operator A. As such the $\mathfrak{b}_\beta$ in (14) are, modulo some integer power of the imaginary unit, derivatives of the symbol of A.

Let us next consider a fixed $u \in \mathbb{R}^d$ and $\eta_u > 0$. Then for each multi-index γ and integer $r \geq 0$ we are going to construct $h_{r,\gamma} = h_{r,\gamma}(u;\eta_u)$ as follows. For $|\gamma| > r$ we set $h_{r,\gamma} \equiv 0$ and for $0 \leq r \leq |\gamma|$, the $h_{r,\gamma}$ are determined by the following recursion. As initialization we take $h_{0,0} \equiv 1$, and for $0 \leq r < |\gamma|$ we have (cf. [1]-(4.6)),

$$(r+1)h_{r+1,\gamma} = \sum_{|\beta| \leq r-|\gamma|} \eta_u^{-1} \binom{\gamma+\beta}{\beta} h_{r,\gamma+\beta} \mathfrak{b}^0_\beta \tag{15}$$
$$+ \sum_{|\kappa|=1,\, \kappa \leq \gamma} \ \sum_{|\beta| \leq r+1-|\gamma|} \eta_u^{-1} \binom{\gamma-\kappa+\beta}{\beta} h_{r,\gamma-\kappa+\beta} \mathfrak{b}^1_{\beta,\kappa} + r h_{r,\gamma},$$

where $|\gamma| \leq r+1$, and empty sums are defined to be zero. We next set

$$h_r(u;\eta_u) := [h_{r,e_i}(u;\eta_u)]_{i=1,\ldots,d}\,.$$

In view of Th. 4.1 in [1] suitable choices of η_u are

$$\eta_u \gtrsim 1 + \|u\|^2 \quad \text{in case of pure affine diffusions,}$$
$$\eta_u \gtrsim e^{\zeta\|u\|}, \ \ \zeta > 0, \ \text{for affine jump processes with thinly tailed large jumps.}$$

In practice the best choice of η_u can be determined in view of the particular problem under consideration. Generally, on the one hand, η_u should be large enough to guarantee convergence of the series (11), but on the other hand should not taken to be unnecessarily large for this would result in series that converges too slowly.

As a natural approximation to (11) and (12) we consider for $K = 1, 2, \ldots,$

$$\ln \widehat{p}_K(t,x,u) = \ln\left(\sum_{r=0}^{K} h_{r,0}(u;\eta_u)(1-e^{-\eta_u t})^r\right) + \mathrm{i}u^\top x \tag{16}$$
$$+ x^\top \frac{\sum_{r=1}^{K} h_r(u;\eta_u)\,(1-e^{-\eta_u t})^r}{\sum_{r=0}^{K} h_{r,0}(u;\eta_u)(1-e^{-\eta_u t})^r}, \quad u \in \mathbb{R}^d, \quad t \geq 0,$$

and the ground expansion based approximation

$$\widehat{p}(t,x,u) = e^{\mathrm{i}u^\top x} \sum_{r=0}^{K} q_r(x,u;\eta_u)(1-e^{-\eta_u t})^r, \quad u \in \mathbb{R}^d, \quad t \geq 0, \tag{17}$$

respectively.

Remark 2 *In connection with approximations (16) and (17) it seems natural to estimate R_u in view of Cauchy's criterion, and η_u according to (10). That is, we could take*

$$\eta_u \approx \frac{\pi}{2} \sqrt[K]{\frac{|A^K f_u(x)|}{K!}},$$

where the sequence $g_r(x,u) := A^r f_u(x)/f_u(x)$ *can be obtained from the recursion*

$$g_{r+1,\gamma} = \sum_{|\beta|\le r-|\gamma|} \binom{\gamma+\beta}{\beta} g_{r,\gamma+\beta}\mathfrak{b}^0_\beta \qquad (18)$$
$$+ \sum_{|\kappa|=1,\,\kappa\le\gamma}\ \sum_{|\beta|\le r+1-|\gamma|} \binom{\gamma-\kappa+\beta}{\beta} g_{r,\gamma-\kappa+\beta}\mathfrak{b}^1_{\beta,\kappa},$$

with $g_{0,0}=1$ *(cf [1]-(4.6)).*

4 Generalized Merton models

We now consider generalized Merton models of the form (5) and (6). For the characteristic function of (5) we have,

$$\Phi_t(z) = e^{\mathrm{i}z\gamma t}\mathbb{E}e^{\mathrm{i}z\sigma W_t}\mathbb{E}e^{\mathrm{i}zX_t^{0;\left(0,x^2,\dots,x^d\right);1}}$$
$$= \exp\left[\mathrm{i}z\gamma t - \frac{z^2\sigma^2}{2}t\right]\widehat{p}(t,\left(0,x^2,...,x^d\right),(z,0,...,0)), \qquad (19)$$

where $X_t^{\cdots;1}$ denotes the first component of $X_t^{\cdots}$ cf. (2). Firstly, the martingale condition (3) can now be formulated as

$$\gamma = -\frac{\sigma^2}{2} - t^{-1}\ln\widehat{p}(t,\left(0,x^2,...,x^d\right),(-\mathrm{i},0,...,0)), \qquad (20)$$

that is, γ may in principle depend on time t. More generally, the characteristic function of (6) takes the form,

$$\Phi_t(z) = e^{\mathrm{i}z\gamma t}\widehat{p}_H(t,z)\widehat{p}(t,\left(0,x^2,...,x^d\right),(z,0,...,0)), \qquad (21)$$

with $\widehat{p}_H(t,z) := \mathbb{E}\left[\exp(\mathrm{i}zH_t)\right]$, and

$$\gamma = -t^{-1}\ln\widehat{p}_H(t,-\mathrm{i}) - t^{-1}\ln\widehat{p}(t,\left(0,x^2,...,x^d\right),(-\mathrm{i},0,...,0)). \qquad (22)$$

In a situation where $\widehat{p}$ in (19) and (21), respectively, is unknown in closed form, we propose to replace it with an approximation $\widehat{p}_K$ due to (16) for some level K large enough. It is convenient to choose X_t^1 and H such that $\exp\left(X_\cdot^1\right)$ and $\exp\left(H_\cdot\right)$ are martingales, respectively. Since $X_0^1 = H_0 = 0$, we then have $\gamma=0$ in (22).

Before considering affine processes with really unknown characteristic function, in the next section we recall the known characteristics of a log-Heston type model.

4.1 The Heston model

Let us consider for X a log-Heston type model with dynamics

$$
\begin{aligned}
dX^1 &= -\frac{1}{2}\alpha^2 X^2 dt + \alpha\sqrt{X^2}dW, \qquad X^1(0) = 0, \\
dX^2 &= \kappa\left(\theta - X^2\right)dt + \sigma\sqrt{X^2}\left(\rho dW + \sqrt{1-\rho^2}d\overline{W}\right), \quad X^2(0) = \theta,
\end{aligned} \tag{23}
$$

for some $\alpha, \sigma, \kappa, \theta > 0$, and $-1 \le \rho \le 1$. Note that the initial value of X^2 is taken to be the expectation of the long-run stationary distribution of X^2. The characteristic function X^1 due to (23) is known as follows (we take Lord and Kahl's representation [6], due to the principal branch of the square root and logarithm[1]):

$$
\ln \widehat{p}(t, \theta, z) := \ln \widehat{p}(t, (0, \theta), (z, 0)) = A(z;t) + B(z;t)\theta, \quad \text{with} \tag{24}
$$

$$
\begin{aligned}
A(z;t) &:= \frac{\theta\kappa}{\sigma^2}\left((a-d)t - 2\ln\frac{e^{-dt}-g}{1-g}\right), \\
B(z;t) &:= \frac{a+d}{\sigma^2}\frac{1-e^{dt}}{1-ge^{dt}} \qquad \text{with} \\
&a := \kappa - \mathfrak{i}z\alpha\sigma\rho, \quad d := \sqrt{a^2 + \alpha^2\sigma^2\left(\mathfrak{i}z + z^2\right)}, \quad g := \frac{a+d}{a-d},
\end{aligned} \tag{25}
$$

while abusing notation in (24) slightly. By construction, $\exp\left(X_t^1\right)$ is a martingale and so it holds that $\ln \widehat{p}(t, \theta, -\mathfrak{i}) = 0$. This can be easily seen from the Heston dynamics (23) and also by taking $z = -\mathfrak{i}$ in (25), where we then have that $a = \kappa - z\alpha\sigma\rho \in \mathbb{R}$, so $d = |a|$. Thus $|g| = \infty$ if $a > 0$ and $|g| = 0$ if $a < 0$ and for both cases we get that $A(-\mathfrak{i};t) \equiv B(-\mathfrak{i};t) \equiv 0$. As a consequence we have $\gamma = -\sigma^2/2$ in (20).

The generator (7) due to the Heston model (23) and its corresponding symbol derivatives (14), i.e. the ingredients of the recursion (15), are spelled out in Appendix A.

4.2 Heston model with state dependent jumps

We now consider a generalized Heston model with state dependent jumps in the first component, henceforth termed the HSDJ model, of the following form:

$$
\begin{aligned}
dX^1 &= -\lambda_0 a_0 dt - \left(\lambda_1 a_1 + \frac{1}{2}\alpha^2\right)X^2 dt + \alpha\sqrt{X^2}dW \\
&\quad + \int_{\mathbb{R}} y\left(N(X^2_-, dt, dy) - \lambda_0\mu_0(y)dydt - X^2\lambda_1\mu_1(y)dydt\right), \\
dX^2 &= \kappa\left(\theta - X^2\right)dt + \sigma\sqrt{X^2}\left(\rho dW + \sqrt{1-\rho^2}d\overline{W}\right)
\end{aligned} \tag{26}
$$

[1] Roger Lord confirmed to J.S. a typo in the published version and so we refer to the preprint version.

with $X^1(0) = 0$, $X^2(0) = \theta$ and with t suppressed in X_{t-} (cf. (8)). In this model $N(w, dt, dy)$ is for each $w > 0$ a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}$ and μ_0 and μ_1 are considered to be probability densities of jumps that arrive at rate $\lambda_0 > 0$ and $w\lambda_1 > 0$, respectively. Further in (26), a_0 and a_1 are non-negative constants given by

$$a_0 = \int (e^y - y - 1)\, \mu_0(y)dy \quad \text{and} \quad a_1 = \int (e^y - y - 1)\, \mu_1(y)dy, \tag{27}$$

hence in particular it is assumed that the measures associated with μ_0 and μ_1 have exponential moments. In the HSDJ model the density μ_0 may have support $\mathbb{R}$, for example Gaussian, while the density μ_1 may be concentrated on $(-\infty, 0)$ for example. In this way λ_0 and μ_0 are responsible for the "normal" random jumps in (26), while λ_1 and μ_1 are responsible for downward jumps which, due to the (state) dependence on X^2, arrive with increasing intensity as the volatility X^2 increases. As such the model covers a stylized empirical fact observed for several underlying quantities, such as assets, indices, or interest rates. Since μ_0 and μ_1 are assumed to be probability densities, the dynamics of X^1 in (26) may also be written as

$$\begin{aligned} dX^1 &= \left(-\lambda_0 \left(\mathfrak{m}_0 + a_0 \right) - \left(\frac{1}{2}\alpha^2 + \lambda_1 \left(\mathfrak{m}_1 + a_1 \right) \right) X^2 \right) dt \\ &\quad + \alpha\sqrt{X^2} dW + \int_{\mathbb{R}} y N(X^2_-, dt, dy), \end{aligned} \tag{28}$$

with

$$\mathfrak{m}_0 := \int_{\mathbb{R}} y\mu_0(y)dy \quad \text{and} \quad \mathfrak{m}_1 := \int_{\mathbb{R}} y\mu_1(y)dy. \tag{29}$$

One can show rigorously that $e^{X_t^1}$ is a martingale with $\mathbb{E}\left[e^{X_t^1}\right] = 1$, and so we may take in (20) $\gamma = -\sigma^2/2$ again. Here we restrict our selves to a heuristic argumentation: From Itô's formula for jump processes we see that for $0 \le u < t$,

$$\begin{aligned} e^{X_t^1} - e^{X_u^1} &= \int_u^t e^{X_{s-}^1} d\left(X_s^1\right)^{\text{cont.}} + \frac{1}{2}\int_u^t e^{X_{s-}^1}\alpha^2 X_s^2 ds + \sum_{u<s\le t} \left\{ e^{X_s^1} - e^{X_{s-}^1} \right\} \\ &= \int_{u+}^t e^{X_{s-}^1} \left(\left(-\lambda_0 \left(\mathfrak{m}_0 + a_0 \right) - \lambda_1 \left(\mathfrak{m}_1 + a_1 \right) X^2 \right) dt + \alpha\sqrt{X_s^2} dW \right) \\ &\quad + \sum_{u<s\le t} \left\{ e^{X_s^1} - e^{X_{s-}^1} \right\}. \end{aligned} \tag{30}$$

So, heuristically, we have that

$$\mathbb{E}\left[\left.\sum_{u<s\leq u+\Delta}\left\{e^{X_s^1}-e^{X_{s-}^1}\right\}\right| X_{u-}\right]\approx e^{X_{u-}^1}\mathbb{E}\left[\left.\sum_{u<s\leq u+\Delta}\left\{e^{X_s^1-X_{s-}^1}-1\right\}\right| X_{u-}\right]$$
$$=e^{X_{u-}^1}\lambda_0\Delta\int\left(e^y-1\right)\mu_0(y)dy+X_{u-}^2e^{X_{u-}^1}\lambda_1\Delta\int\left(e^y-1\right)\mu_1(y)dy$$
$$=e^{X_{u-}^1}\left(\lambda_0\left(\mathfrak{m}_0+a_0\right)+X_{u-}^2\lambda_1\left(\mathfrak{m}_1+a_1\right)\right)\Delta,$$

for $\Delta\downarrow 0$. Combining with (30) this yields

$$e^{X_{u+\Delta}^1}-e^{X_u^1}\approx e^{X_{u-}^1}\alpha\sqrt{X_u^2}\Delta W+\zeta_{u,u+\Delta}$$

with $\mathbb{E}\left[\zeta_{u,u+\Delta}\middle| X_{u-}\right]=0$.

In Appendix B we spell out the generator, cf. (7), and its corresponding symbol derivatives (14) corresponding to the HSDJ model (26).

Example 3 *In the case where $\lambda_1=0$, the characteristic function $\widehat{p}_{\lambda_0,\mu_0}$ of X^1 is simply given by (see (28), (27) and (29))*

$$\ln\widehat{p}_{\lambda_0,\mu_0}(t,\theta,z)=\ln\widehat{p}(t,\theta,z)-t\lambda_0\left(a_0+\mathfrak{m}_0\right)\mathrm{i}z+t\lambda_0\psi_0(z)$$
$$=\ln\widehat{p}(t,\theta,z)-t\lambda_0\psi_0(-\mathrm{i})\mathrm{i}z+t\lambda_0\psi_0(z),$$

where

$$\psi_0(z):=\int\left(e^{\mathrm{i}yz}-1\right)\mu_0(y)dy=\int e^{\mathrm{i}yz}\mu_0(y)dy-1$$

follows from the characteristic function of the jump measure and $\widehat{p}(t,\theta,z)$ is given by (24). Note that we have $\ln\widehat{p}_{\lambda_0,\mu_0}(t,\theta,-\mathrm{i})=0$ again indeed. For example if the jumps are $\mathcal{N}(c,\nu^2)$ distributed we have the well known expression

$$\psi_0(z)=e^{\mathrm{i}cz-\frac{1}{2}\nu^2z^2}-1,$$

hence

$$\ln\widehat{p}_{\lambda_0,c,\nu^2}(t,\theta,z):=\ln\widehat{p}(t,\theta,z)+t\lambda_0\left(e^{\mathrm{i}cz-\frac{1}{2}\nu^2z^2}-\mathrm{i}ze^{c+\frac{1}{2}\nu^2}+\mathrm{i}z-1\right).$$

5 Numerical examples

In this section we will price European options by a Fourier based method due to Carr-Madan [2]. Let the stock price at maturity T be given as

$$S_T=S_0e^{rT+Y_T},$$

where $\exp\left[Y_\cdot\right]$ is a martingale with $Y_0=0$. If the characteristic function

$$\Phi_T(z):=\mathbb{E}\left[e^{\mathrm{i}zY_T}\right]$$

is known, then the the price of a European call option with strike K at time $t = 0$ is given by

$$C(K) = (S_0 - Ke^{-rT})^+ + \frac{S_0}{2\pi}\int_{-\infty}^{\infty} \frac{1 - \Phi_T(z - \mathrm{i})}{z(z - \mathrm{i})} e^{-\mathrm{i}z \ln \frac{Ke^{-rT}}{S_0}} dz \tag{31}$$

(Carr-Madan's formula). For more general Fourier valuation formulas, see [5]. In general, the decay of the integrand in (31) is of order $O(|z|^{-2})$ as $|z| \to \infty$, hence relatively slow. We therefore use a kind of variance reduction for integrals using the formula

$$\mathcal{BS}\,(S_0, T, r, \sigma_B) = (S_0 - Ke^{-rT})^+ + \frac{S_0}{2\pi}\int_{-\infty}^{\infty} \frac{1 - \Phi_T^{\mathcal{BS}}(z - \mathrm{i})}{z(z - \mathrm{i})} e^{-\mathrm{i}z \ln \frac{Ke^{-rT}}{S_0}} dz, \tag{32}$$

where $\mathcal{BS}$ is the well-known Black-Scholes formula based on the risk-neutral Black-Scholes model

$$S_t^{\mathcal{B}} := S_0 e^{rT - \sigma_B^2 T/2 + \sigma_B W_T}, \quad \text{with}$$
$$\Phi_T^{\mathcal{BS}}(z) := \mathbb{E}\left[e^{\mathrm{i}z\left(-\sigma_B^2 T/2 + \sigma_B W_T\right)}\right] = e^{-\left(z^2 + \mathrm{i}z\right)\sigma_B^2 T/2},$$

for a suitable but in principle arbitrary $\sigma_B > 0$. Next, subtracting (31) and (32) gives the variance reduced formula

$$C(K) = \mathcal{BS}\,(S_0, T, r, \sigma_B) + \frac{S_0}{2\pi}\int_{-\infty}^{\infty} \frac{\Phi_T^{\mathcal{BS}}(z - \mathrm{i}) - \Phi_T(z - \mathrm{i})}{z(z - \mathrm{i})} e^{-\mathrm{i}z \ln \frac{Ke^{-rT}}{S_0}} dz, \tag{33}$$

where, typically, the integrand decays much faster than in (31).

5.1 Product of Heston models

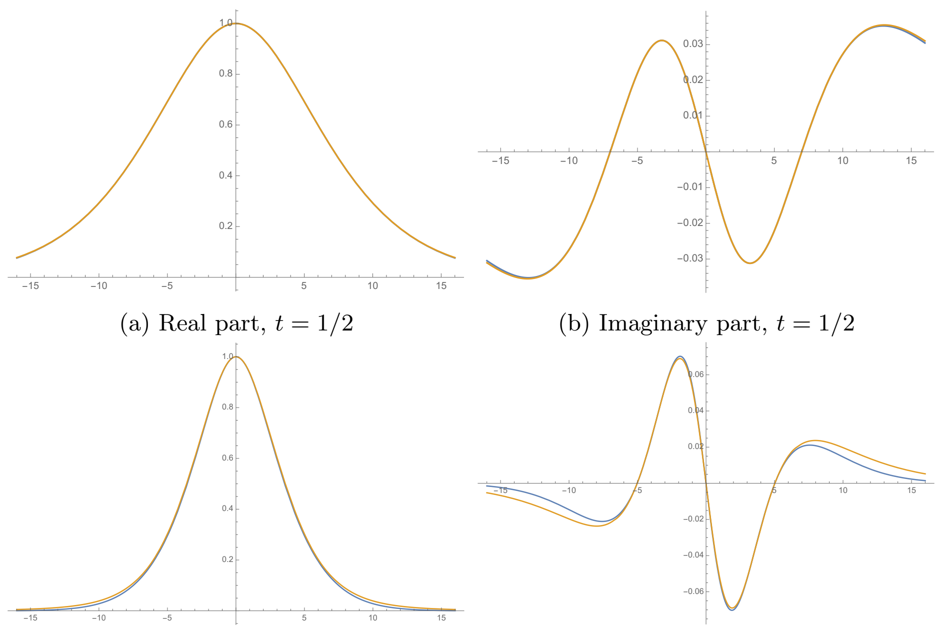
We first consider a model where the stock price S_t is obtained as the product of two independent Heston factors, i.e., (6) with X_t^1 another Heston model. Clearly, in this case a closed form expression for the characteristic function of $\ln S_t$ exists, and therefore the asymptotic expansion presented in this paper is not needed for pricing. This allows us to easily compute accurate reference prices, and thus assess the numerical accuracy of prices obtained from the expansion of the characteristic function. All calculations were done using Mathematica. Using its symbolic capabilities, we have implemented the recursion (15) in full generality.

The Heston parameters for the components H_t and X_t^1 are presented in Table1. Additionally, we choose $S_0 = 10$ and $r = 0.05$ for option pricing. Based on these parameters, we compute the asymptotic expansion $\widehat{p}_K$ of the characteristic function using (12) with $K = 8$, i.e., including the first *nine* terms in the expansion.

In Figure 1, we compare the exact and the approximate characteristic functions of the (normalized) logarithm of the stock prices—i.e., with $S_0 = 1$ for

	H_t	X_t^1
α	1.0	1.0
κ	1.5	1.5
σ	0.6	0.3
θ	0.04	0.0225
ρ	-0.2	-0.3
v	0.04	0.0225

Table 1: Parameters of the Heston+Heston-model. v denotes the initial variance in both components.



(a) Real part, $t = 1/2$

(b) Imaginary part, $t = 1/2$

(c) Real part, $t = 2$

(d) Imaginary part, $t = 2$

Figure 1: Exact (blue) and approximate (orange) characteristic functions of the logarithm of the normalized stock price in the generalized Merton model with two Heston factors evaluated at time $t = 1/2$ and $t = 2$ (years).

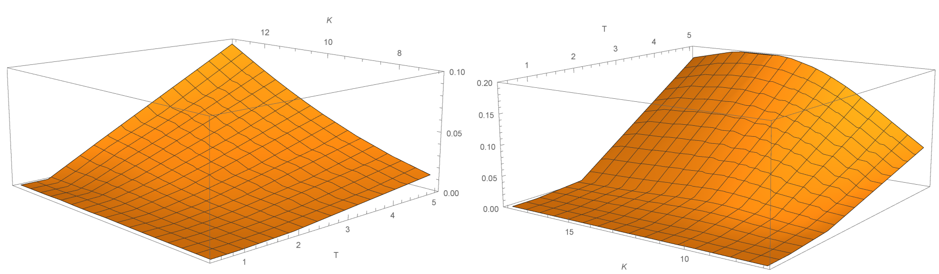
convenience. We can clearly see that the approximation deteriorates when $|u|$ becomes large, but then both the exact and the approximate characteristic functions tend to 0. Moreover, the approximation formula is more accurate for small t.

L	2	4	8	16	32	64
Exact	0.8350	0.9621	1.1105	1.1832	1.1884	1.884
Approx.	0.8353	0.9626	1.1111	1.1842	1.1896	1.1896
(Rel. error)	0.2981	0.1912	0.0665	0.0054	0.0010	0.0010

Table 2: Price of ATM call option with maturity $T = 1$ computed using domain of integration $[-L, L]$ for both the exact characteristic function and the approximate formula, together with the relative error for using the approximate formula—w.r.t. the most accurate price obtained from the exact formula.

When we come to option pricing, we plug the approximate formula for the characteristic function into the Fourier pricing formula (33). For the implementation, we clearly need to replace the infinite domain of integration by a finite one, i.e., we use (33) integrating from $-L$ to L, $L \in \mathbb{R}$. This cut-off is potentially critical for our approximation procedure, as large integration domains (and, hence, large $|u|$) may correspond to large errors of the approximate formula. Fortunately, Table 2 indicates that this effect does not materialize.

Remark 4 *At this stage, we would like to highlight once more the heuristic choice of η proposed in Remark 2. Without a good choice of η, it is very easy to run into situations, where the approximation error is already too large for the needed domain of integration.*



(a) Relative error.

(b) Absolute error.

Figure 2: Relative and absolute errors of European call option prices.

Let us consider option prices and the corresponding errors for maturities from 1/2 to 5 years and for strike prices between 7 (deep in) and 13 (deep out of) the money. Figure 2 shows that errors remain small ($\leq 2\%$ ATM) for maturities up to 2 years. For (deep) OTM options, it seems to be more reasonable to look at absolute instead of relative errors, which give a similar impression.

Finally, the implied volatility in this model is plotted in Figure 3. Considerable deviations between the exact and the approximate formula are only observed for higher maturities.

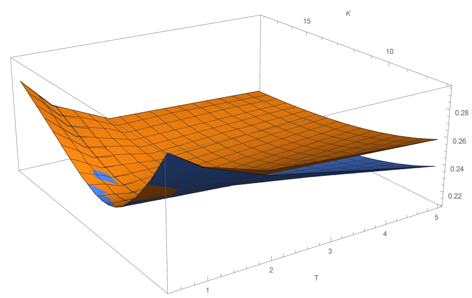


Figure 3: Implied volatility of the generalized Merton model based on two Heston factors based on exact (blue) and approximate (orange) characteristic functions.

5.2 Generalized Merton model with state-dependent jumps

Let us consider a generalized Merton model of the form (6) where X^1 is an affine jump process with state-dependent jump-intensity In the sense of (26). The parameters corresponding to the diffusive parts of both H and X^1 are chosen as in Table 1. Regarding the jump part of X^1, we set $\lambda_0 = 0$, $\mu_0 = 0$, thereby turning off the jumps with constant, i.e., not state dependent, intensities. The jump parameters of X^1 are chosen according to Table3.

	X_t^1
λ_1	10
$\mu_1(y)$	$\mathbf{1}_{y<0} p e^{py}$
p	4.48

Table 3: Jump parameters of X^1

This means that jumps in the log-price have exponentially distributed magnitude and negative sign. The mean jump of the log-price is around 0.22, i.e., in case of a downward jump ("crisis"), the stock loses about 20% of its value. The intensity λ_1 seems excessively high, but recall that this intensity is multiplied by the instantaneous variance of the Heston component, which is started at 0.04.

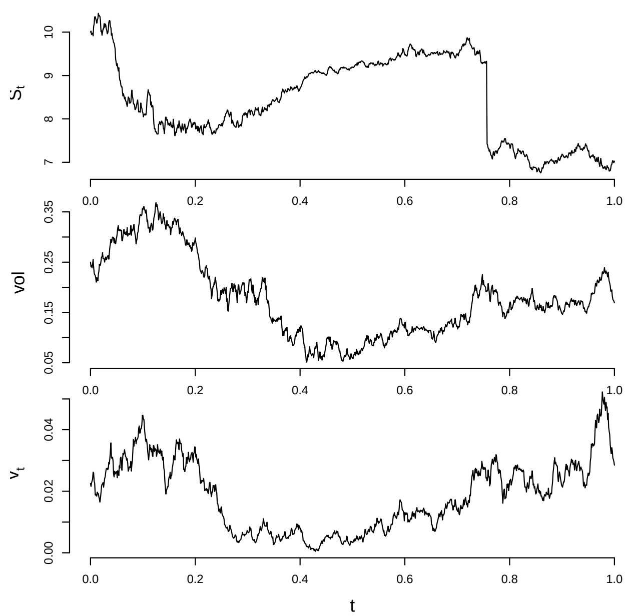
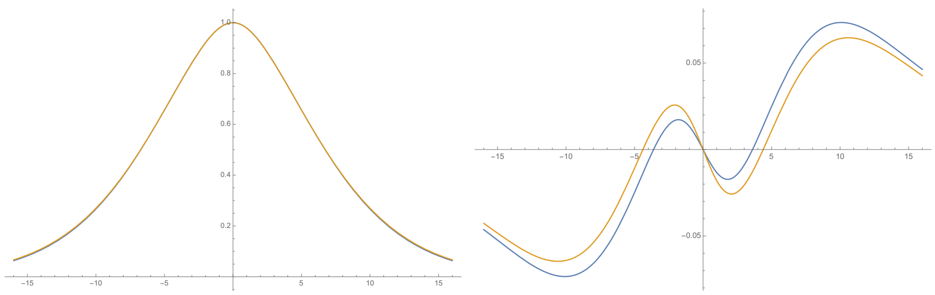


Figure 4: Sample path of S_t in the generalized Merton model with state-dependent jumps (first panel), volatility (more precisely, the square root of the sum of both variance components) of S_t (second panel), and of the variance component of the second Heston factor. A jump occurs shortly after time 0.75.

By (34), and (35) below, we obtain

$$\psi_0(\xi) = 0, \qquad \mathfrak{m}_0 + a_0 = \psi_0(-\mathfrak{i}) = 0,$$

$$\psi_1(\xi) = \int_{-\infty}^{0} \left(e^{\mathfrak{i}\xi y} - 1\right) p e^{py} dy = -\frac{\mathfrak{i}\xi}{p + \mathfrak{i}\xi}, \qquad \mathfrak{m}_1 + a_1 = \psi_1(-\mathfrak{i}) = -\frac{1}{p+1}.$$



(a) Real part, $t = 1$ (b) Imaginary part, $t = 1$

Figure 5: Approximate characteristic function (orange) of the logarithm of the normalized stock price in the generalized Merton model with one Heston factor and one Heston factor with jumps evaluated at time $t = 1/2$ (year). Comparison with the characteristic function computed by a Monte Carlo simulation (blue).

Figure 5 shows the approximate characteristic function including jumps at time $t = 1/2$, compared with the exact characteristic function without jumps. As expected, the jumps lead to a considerable change in the characteristic function. We compare the characteristic function to another numerical approximation based on Monte Carlo simulation. Both approximations lead to very close results especially in the real part. The results are less close for the imaginary part, but notice that the graphical representation exaggerates the differences as the scale is much smaller in the second plot (from -0.1 to 0.1 instead of 0 to 1).

These changes in the distribution have the expected changes in the option prices. In particular, the implied volatilities become larger, and also the smile becomes much more pronounced, comparing Figure 6 with Figure 3.

K	7	8	9	10	11	12	13
Monte Carlo	3.2719	2.3688	1.5511	0.8888	0.4427	0.2006	0.0884
Asym. formula	3.2279	2.3276	1.5144	0.8583	0.4217	0.1880	0.0818
Rel. error	0.0134	0.0174	0.0237	0.0343	0.0476	0.0627	0.0744
$\frac{\text{MC stat. error}}{\text{Ref. price}}$	0.0018	0.0023	0.0031	0.0044	0.0067	0.0106	0.0166

Table 4: Option prices for maturity $T = 1/2$ for various strike prices in the Heston model plus jumps. We compare prices obtained by the asymptotic expansion of the characteristic function with prices obtained by Monte Carlo simulation.

Finally, let us directly compare the price for some European call options with reference prices obtained by Monte Carlo simulation, see Table 4. Once again,

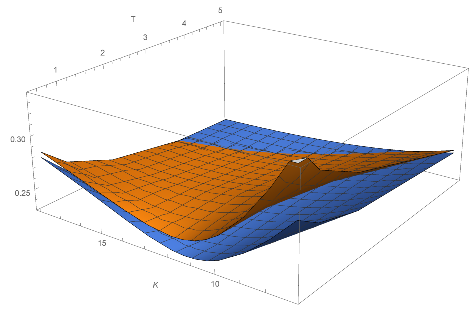


Figure 6: Implied volatility of the generalized Merton model with one Heston factor and one Heston factor with jumps (orange), compared with the implied volatilities computed with the exact characteristic function in Figure 3.

we used $S_0 = 10$ and $r = 0.05$. The Monte Carlo prices are based on $100,000$ trajectories with 1000 time-steps each, the statistical error, i.e., the standard deviation divided by the square root of the number of samples, is considerable smaller than the observed difference.

Unfortunately, the results of Table 4 are not as convincing as the accuracy of the approximation in the pure diffusion case suggested, compare Table 2 and Figure 2. We suspect a combination of slow decay of the characteristic function, sub-optimal choice of the damping parameter η and higher truncation error of the asymptotic characteristic function, see the conclusions below for some further comments.

Conclusions From the examples we conclude that for times being not too large the approximation procedure based on [1] performs rather well. More specifically, if no jumps are in the play the procedure works very good, but with incorporated (state dependent) jumps the accuracy is somewhat less. In order to resolve this issue one could investigate different directions. One reason for less accuracy may be a diminished effect of the Black-Scholes ingredients in the Fourier pricing formula (33) in the presence of state dependent jumps. This in turn might require a larger integration range where that approximation gets worse at the upper and lower end, respectively. As a way out, it looks natural to replace the role of the Black-Scholes ingredients in (33) by an affine model with state independent jumps for which the characteristic function is known, leading to a representation of the form

$$C^{\mathrm{appr}}(K) = (S_0 - Ke^{-rT})^+ + \frac{S_0}{2\pi}\int_{-\infty}^{\infty} \frac{1 - \Phi_T^{\mathrm{known}}(z - \mathfrak{i})}{z(z - \mathfrak{i})} e^{-\mathfrak{i}z \ln \frac{Ke^{-rT}}{S_0}} dz$$
$$+ \frac{S_0}{2\pi}\int_{-\infty}^{\infty} \frac{\Phi_T^{\mathrm{known}}(z - \mathfrak{i}) - \Phi_T^{\mathrm{appr}}(z - \mathfrak{i})}{z(z - \mathfrak{i})} e^{-\mathfrak{i}z \ln \frac{Ke^{-rT}}{S_0}} dz =: I_{\mathrm{known}} + I_{\mathrm{appr}}.$$

The integral I_{known} can be computed with any desired accuracy while for the integral I_{appr} a relatively small integration range may be sufficient.

Other reasons for the decreased accuracy in Section 5.2 for instance, may be a too small η chosen due to Remark 2, or not enough iterations. However, we leave all these investigations for further research, since this article is considered merely a first guide on numerical implementation of the method in [1].

A Generator and $\mathfrak{b}_\beta$ for the Heston model

By conferring (7), (8), and (23), the generator of the Heston model is given by

$$A = -\frac{1}{2}\alpha^2 x_2 \partial_{x_1} + \kappa\,(\theta - x_2)\,\partial_{x_2} + \frac{1}{2}\alpha^2 x_2 \partial_{x_1 x_1} + \alpha\sigma\rho x_2 \partial_{x_1 x_2} + \frac{1}{2}\sigma^2 x_2 \partial_{x_2 x_2}.$$

It thus follows with $f_u(x) = e^{\mathrm{i}u^\top x}$ that

$$\frac{Af_u(x)}{f_u(x)} = -\frac{1}{2}\alpha^2 x_2 \mathrm{i}u_1 + \kappa\,(\theta - x_2)\,\mathrm{i}u_2 - \frac{1}{2}\alpha^2 x_2 u_1^2 - \sigma\alpha\rho x_2 u_1 u_2 - \frac{1}{2}\sigma^2 x_2 u_2^2$$

with first order derivatives w.r.t. u,

$$\begin{aligned}
\partial_{u_1}\frac{Af_u(x)}{f_u(x)} &= -\frac{1}{2}\alpha^2 x_2 \mathrm{i} - \alpha^2 x_2 u_1 - \alpha\sigma\rho x_2 u_2,\\
\partial_{u_2}\frac{Af_u(x)}{f_u(x)} &= \kappa\,(\theta - x_2)\,\mathrm{i} - \alpha\sigma\rho x_2 u_1 - \sigma^2 x_2 u_2.
\end{aligned}$$

For the second derivatives we get

$$\partial_{u_1 u_1}\frac{Af_u(x)}{f_u(x)} = -\alpha^2 x_2, \quad \partial_{u_2 u_2}\frac{Ae^{\mathrm{i}u^\top x}}{e^{\mathrm{i}u^\top x}} = -\sigma^2 x_2, \quad \partial_{u_1 u_2}\frac{Ae^{\mathrm{i}u^\top x}}{e^{\mathrm{i}u^\top x}} = -\alpha\sigma\rho x_2,$$

and the third order ones vanish. Thus, in multi-index notation we have by (14) for $|\beta| = 0$,

$$\mathfrak{b}_0(x,u) = \kappa\theta\mathrm{i}u_2 + x_2\left(-\frac{1}{2}\alpha^2\mathrm{i}u_1 - \kappa\mathrm{i}u_2 - \frac{1}{2}\alpha^2 u_1^2 - \alpha\sigma\rho u_1 u_2 - \frac{1}{2}\sigma^2 u_2^2\right),$$

whence

$$\begin{aligned}
&\mathfrak{b}_0^0(u) = \kappa\theta\mathrm{i}u_2,\\
&\mathfrak{b}_{0,e_1}^1(u) = 0, \quad \mathfrak{b}_{0,e_2}^1(u) = -\frac{1}{2}\alpha^2\mathrm{i}u_1 - \kappa\mathrm{i}u_2 - \frac{1}{2}\alpha^2 u_1^2 - \alpha\sigma\rho u_1 u_2 - \frac{1}{2}\sigma^2 u_2^2
\end{aligned}$$

with $e_1 := (1,0)$, $e_2 := (0,1)$. For $|\beta| = 1$, (14) yields

$$\begin{aligned}
\mathfrak{b}_{(1,0)}(x,u) &= -\frac{1}{2}\alpha^2 x_2 + \alpha^2 x_2 u_1 \mathrm{i} + \alpha\sigma\rho x_2 u_2 \mathrm{i},\\
\mathfrak{b}_{(0,1)}(x,u) &= \kappa\,(\theta - x_2) + \alpha\sigma\rho x_2 u_1 \mathrm{i} + \sigma^2 x_2 u_2 \mathrm{i},
\end{aligned}$$

whence

$$\mathfrak{b}^0_{(1,0)}(u) = \mathfrak{b}^1_{(1,0),e_1}(u) = 0, \quad \mathfrak{b}^1_{(1,0),e_2}(u) = -\frac{1}{2}\alpha^2 + \alpha^2 u_1 \mathfrak{i} + \alpha\sigma\rho u_2 \mathfrak{i}$$

and

$$\mathfrak{b}^0_{(0,1)}(u) = \kappa\theta, \quad \mathfrak{b}^1_{(0,1),e_1}(u) = 0, \quad \mathfrak{b}^1_{(0,1),e_2}(u) = -\kappa + \alpha\sigma\rho u_1 \mathfrak{i} + \sigma^2 u_2 \mathfrak{i}$$

Next, for $|\beta| = 2$, (14) yields

$$\mathfrak{b}_{(2,0)}(x,u) = \alpha^2 x_2, \quad \mathfrak{b}_{(0,2)}(x,u) = \sigma^2 x_2, \quad \mathfrak{b}_{(1,1)}(x,u) = \alpha\sigma\rho x_2,$$

whence

$$\begin{aligned}
\mathfrak{b}^0_{(2,0)}(u) &= \mathfrak{b}^1_{(2,0),e_1}(u) = 0, \quad \mathfrak{b}^1_{(2,0),e_2}(u) = \alpha^2, \\
\mathfrak{b}^0_{(0,2)}(u) &= \mathfrak{b}^1_{(0,2),e_1}(u) = 0, \quad \mathfrak{b}^1_{(0,2),e_2}(u) = \sigma^2, \\
\mathfrak{b}^0_{(1,1)}(u) &= \mathfrak{b}^1_{(1,1),e_1}(u) = 0, \quad \mathfrak{b}^1_{(1,1),e_2}(u) = \alpha\sigma\rho,
\end{aligned}$$

and for $|\beta| \geq 3$, we trivially find

$$\mathfrak{b}_\beta(x,u) = 0.$$

B Generator and $\mathfrak{b}_\beta$ for the HSDJ model

By conferring (7), (8), and (26), we have in fact

$$v(x,dz) = v^0(dz) + x^\top v^1(dz) = \lambda_0\mu_0(z_1)\delta_0(z_2)dz_1dz_2 + x_2\lambda_1\mu_1(z_1)\delta_0(z_2)dz_1dz_2$$

with δ_0 being the Dirac delta function, that is the (singular) density of the Dirac probability measure $\mathbb{R}$ concentrated in $\{0\}$. Thus, the generator of the HSDJ model is given by

$$\begin{aligned}
Af(x_1,x_2) = {}& \left(-\lambda_0 a_0 - \left(\frac{1}{2}\alpha^2 + \lambda_1 a_1\right) x_2\right) \partial_{x_1} f + \kappa(\theta - x_2)\partial_{x_2} f \\
& + \frac{1}{2}\alpha^2 x_2 \partial_{x_1x_1} f + \alpha\sigma\rho x_2 \partial_{x_1x_2} f + \frac{1}{2}\sigma^2 x_2 \partial_{x_2x_2} f \\
& + \int_{\mathbb{R}} \left[f(x_1+z_1,x_2) - f(x_1,x_2) - z_1\partial_{x_1} f\right] \left(\lambda_0\mu_0(z_1)dz_1 + x_2\lambda_1\mu_1(z_1)dz_1\right).
\end{aligned}$$

Since we are dealing with jump probability densities rather than infinite jump measures, as in the case of infinite activity processes, the generator may be written as

$$\begin{aligned}
Af(x_1,x_2) = {}& \left(-\lambda_0(\mathfrak{m}_0 + a_0) - \left(\frac{1}{2}\alpha^2 + \lambda_1(\mathfrak{m}_1 + a_1)\right) x_2\right) \partial_{x_1} f \\
& + \kappa(\theta - x_2)\partial_{x_2} f + \frac{1}{2}\alpha^2 x_2 \partial_{x_1x_1} f + \alpha\sigma\rho x_2 \partial_{x_1x_2} f + \frac{1}{2}\sigma^2 x_2 \partial_{x_2x_2} f \\
& + \lambda_0 \int_{\mathbb{R}} \left[f(x_1+y,x_2) - f(x_1,x_2)\right] \mu_0(y)dy \\
& + x_2\lambda_1 \int_{\mathbb{R}} \left[f(x_1+y,x_2) - f(x_1,x_2)\right] \mu_1(y)dy,
\end{aligned}$$

using (29).

With $f_u(x) = e^{\mathrm{i}u^\top x}$ we so obtain,

$$\begin{aligned}\frac{Af_u(x)}{f_u(x)} &= \left(-\lambda_0\left(\mathfrak{m}_0+a_0\right)-\left(\frac{1}{2}\alpha^2+\lambda_1\left(\mathfrak{m}_1+a_1\right)\right)x_2\right)\mathrm{i}u_1 \\ &+\kappa\left(\theta-x_2\right)\mathrm{i}u_2-\frac{1}{2}\alpha^2x_2u_1^2-\alpha\sigma\rho x_2u_1u_2-\frac{1}{2}\sigma^2x_2u_2^2 \\ &+\lambda_0\psi_0(u_1)+x_2\lambda_1\psi_1(u_1)\end{aligned}$$

with

$$\psi_i(\xi) := \int_{\mathbb{R}}\left(e^{\mathrm{i}\xi y}-1\right)\mu_i(y)dy, \quad i=0,1. \tag{34}$$

Note that we have

$$\mathfrak{m}_i + a_i = \psi_i(-\mathrm{i}), \quad i=0,1. \tag{35}$$

The first order derivatives w.r.t. u are,

$$\begin{aligned}\partial_{u_1}\frac{Af_u(x)}{f_u(x)} &= -\lambda_0\left(\mathfrak{m}_0+a_0\right)\mathrm{i}-\left(\frac{1}{2}\alpha^2+\lambda_1\left(\mathfrak{m}_1+a_1\right)\right)\mathrm{i}x_2 \\ &-\alpha^2x_2u_1-\alpha\sigma\rho x_2u_2+\lambda_0\partial_{u_1}\psi_0(u_1)+x_2\lambda_1\partial_{u_1}\psi_1(u_1) \\ \partial_{u_2}\frac{Af_u(x)}{f_u(x)} &= \kappa\left(\theta-x_2\right)\mathrm{i}-\alpha\sigma\rho x_2u_1-\sigma^2x_2u_2.\end{aligned}$$

For the second order derivatives we have

$$\begin{aligned}\partial_{u_1u_1}\frac{Af_u(x)}{f_u(x)} &= -\alpha^2x_2+\lambda_0\partial_{u_1u_1}\psi_0(u_1)+x_2\lambda_1\partial_{u_1u_1}\psi_1(u_1) \\ \partial_{u_1u_2}\frac{Af_u(x)}{f_u(x)} &= -\alpha\sigma\rho x_2, \quad \partial_{u_2u_2}\frac{Af_u(x)}{f_u(x)} = -\sigma^2x_2,\end{aligned}$$

and for multi-indices β with $|\beta| \geq 3$, i.e. the higher order ones,

$$\partial_{u^\beta}\frac{Af_u(x)}{f_u(x)} = \begin{cases}\lambda_0\partial_{u_1^{|\beta|}}\psi_0(u_1)+x_2\lambda_1\partial_{u_1^{|\beta|}}\psi_1(u_1) & \text{for} \quad \beta=(|\beta|,0), \\ 0 \quad \text{if} \quad \beta\neq(|\beta|,0).\end{cases}$$

Hence the ingredients (14) of the recursion (15) are in multi-index notation as follows.

$|\beta| = 0:$

$$\begin{aligned}\mathfrak{b}_0(x,u) &= -\lambda_0\left(\mathfrak{m}_0+a_0\right)\mathrm{i}u_1+\kappa\theta\mathrm{i}u_2+\lambda_0\psi_0(u_1) \\ +x_2&\left(\lambda_1\psi_1(u_1)-\left(\frac{1}{2}\alpha^2+\lambda_1\left(\mathfrak{m}_1+a_1\right)\right)\mathrm{i}u_1-\kappa\mathrm{i}u_2-\frac{1}{2}\alpha^2u_1^2-\alpha\sigma\rho u_1u_2-\frac{1}{2}\sigma^2u_2^2\right)\end{aligned}$$

whence

$$\begin{aligned}\mathfrak{b}_0^0(u) &= -\lambda_0\left(\mathfrak{m}_0+a_0\right)\mathrm{i}u_1+\kappa\theta\mathrm{i}u_2+\lambda_0\psi_0(u_1), \\ \mathfrak{b}_{0,e_1}^1(u) &= 0, \quad \mathfrak{b}_{0,e_2}^1(u) = \lambda_1\psi_1(u_1)-\left(\frac{1}{2}\alpha^2+\lambda_1\left(\mathfrak{m}_1+a_1\right)\right)\mathrm{i}u_1 \\ &-\kappa\mathrm{i}u_2-\frac{1}{2}\alpha^2u_1^2-\alpha\sigma\rho u_1u_2-\frac{1}{2}\sigma^2u_2^2.\end{aligned}$$

For $|\beta| = 1$, (14) yields

$$\begin{aligned}
\mathfrak{b}_{(1,0)}(x,u) &= -\lambda_0 (\mathfrak{m}_0 + a_0) - \lambda_0 \partial_{u_1} \psi_0(u_1)\mathfrak{i} - \left(\frac{1}{2}\alpha^2 + \lambda_1 (\mathfrak{m}_1 + a_1)\right) x_2 \\
&\quad + \alpha^2 x_2 u_1 \mathfrak{i} + \alpha\sigma\rho x_2 u_2 \mathfrak{i} - x_2 \lambda_1 \partial_{u_1} \psi_1(u_1)\mathfrak{i} \\
\mathfrak{b}_{(0,1)}(x,u) &= \kappa (\theta - x_2) + \alpha\sigma\rho x_2 u_1 \mathfrak{i} + \sigma^2 x_2 u_2 \mathfrak{i},
\end{aligned}$$

whence

$$\begin{aligned}
\mathfrak{b}^0_{(1,0)}(u) &= -\lambda_0 (\mathfrak{m}_0 + a_0) - \lambda_0 \partial_{u_1} \psi_0(u_1)\mathfrak{i}, \quad \mathfrak{b}^1_{(1,0),e_1}(u) = 0, \\
\mathfrak{b}^1_{(1,0),e_2}(u) &= -\left(\frac{1}{2}\alpha^2 + \lambda_1 (\mathfrak{m}_1 + a_1)\right) + \alpha^2 u_1 \mathfrak{i} + \alpha\sigma\rho u_2 \mathfrak{i} - \lambda_1 \partial_{u_1} \psi_1(u_1)\mathfrak{i}
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{b}^0_{(0,1)}(u) &= \kappa\theta, \quad \mathfrak{b}^1_{(0,1),e_1}(u) = 0, \\
\mathfrak{b}^1_{(0,1),e_2}(u) &= -\kappa + \alpha\sigma\rho u_1 \mathfrak{i} + \sigma^2 u_2 \mathfrak{i}.
\end{aligned}$$

Next, for $|\beta| = 2$, (14) yields

$$\begin{aligned}
\mathfrak{b}_{(2,0)}(x,u) &= \alpha^2 x_2 - \lambda_0 \partial_{u_1 u_1} \psi_0(u_1) - x_2 \lambda_1 \partial_{u_1 u_1} \psi_1(u_1), \\
\mathfrak{b}_{(1,1)}(x,u) &= \alpha\sigma\rho x_2, \\
\mathfrak{b}_{(0,2)}(x,u) &= \sigma^2 x_2
\end{aligned}$$

whence

$$\begin{aligned}
\mathfrak{b}^0_{(2,0)}(u) &= -\lambda_0 \partial_{u_1 u_1} \psi_0(u_1), \quad \mathfrak{b}^1_{(2,0),e_1}(u) = 0, \\
\mathfrak{b}^1_{(2,0),e_2}(u) &= \alpha^2 - \lambda_1 \partial_{u_1 u_1} \psi_1(u_1), \\
\mathfrak{b}^0_{(1,1)}(u) &= \mathfrak{b}^1_{(1,1),e_1}(u) = 0, \quad \mathfrak{b}^1_{(1,1),e_2}(u) = \alpha\sigma\rho, \\
\mathfrak{b}^0_{(0,2)}(u) &= \mathfrak{b}^1_{(0,2),e_1}(u) = 0, \quad \mathfrak{b}^1_{(0,2),e_2}(u) = \sigma^2.
\end{aligned}$$

For multi-indices β with $|\beta| \geq 3$ we get

$$\mathfrak{b}_\beta(x,u) = \begin{cases} \lambda_0 \mathfrak{i}^{-|\beta|} \partial_{u_1^{|\beta|}} \psi_0(u_1) + x_2 \lambda_1 \mathfrak{i}^{-|\beta|} \partial_{u_1^{|\beta|}} \psi_1(u_1) \quad \text{for} \quad \beta = (|\beta|, 0), \\ 0 \quad \text{if} \quad \beta \neq (|\beta|, 0), \end{cases}$$

whence

$$\mathfrak{b}^0_\beta(u) = \begin{cases} \lambda_0 \mathfrak{i}^{-|\beta|} \partial_{u_1^{|\beta|}} \psi_0(u_1) \quad \text{for} \quad \beta = (|\beta|, 0), \\ 0 \quad \text{if} \quad \beta \neq (|\beta|, 0), \end{cases}$$

and

$$\begin{aligned}
\mathfrak{b}^1_{\beta,e_1}(u) &= 0, \\
\mathfrak{b}^1_{\beta,e_2}(u) &= \begin{cases} \lambda_1 \mathfrak{i}^{-|\beta|} \partial_{u_1^{|\beta|}} \psi_1(u_1) \quad \text{for} \quad \beta = (|\beta|, 0), \\ 0 \quad \text{if} \quad \beta \neq (|\beta|, 0). \end{cases}
\end{aligned}$$

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