W I A S

Weierstraß-Institut für Angewandte Analysis und Stochastik

3D Boundary Conforming Delaunay Mesh Generation

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Weierstrass Institute for Applied Analysis and Stochastics
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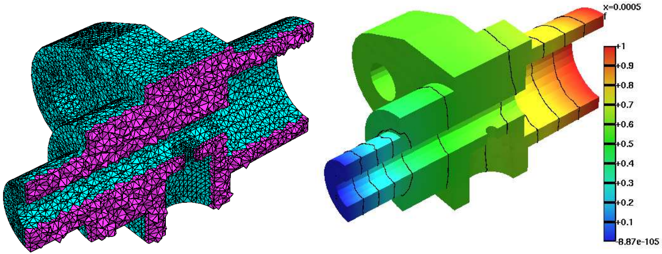
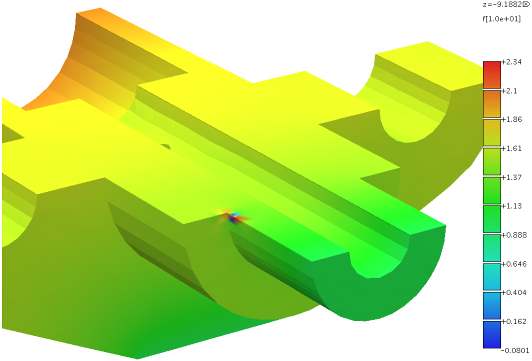
Motivation

- Using numerical methods (such as finite element and finite volume methods) to solve partial differential equations.
- The simulation domain Ω must be subdivided into many simple cells – mesh generation.
- This talk focuses on tetrahedral mesh generation for $\Omega \in \mathbb{R}^3$.



A tetrahedral mesh and the numerical solution of a heat equation.

Motivation



A wrong solution caused by a bad-quality and non-Delaunay mesh.

Motivation

What is a "good" quality mesh?

- Problem-dependent: isotropic, anisotropic, etc.
- Method-dependent: finite element, finite volume, etc.

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- ▷ Problem-dependent: isotropic, anisotropic, etc.
- ▷ Method-dependent: finite element, finite volume, etc.

How to efficiently generate it?

- ▷ Guarantee the quality theoretically.
- ▷ Complete it in polynomial time.

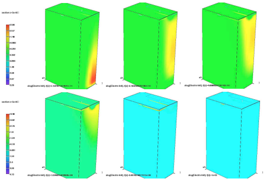
Outline

1. Introduction
2. Delaunay Refinement
3. Adaptive Refinement and Coarsening
4. Application Examples
5. Conclusion

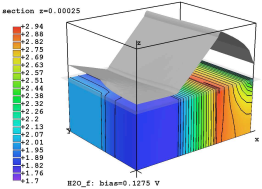
Outline

FVM is a discretization method well suited for numerical simulation of PDEs.

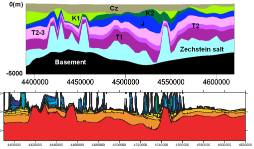
Semiconductor devices



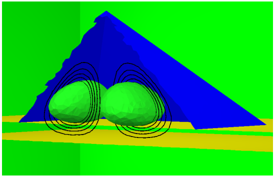
Fuel cells



Thermohaline convection

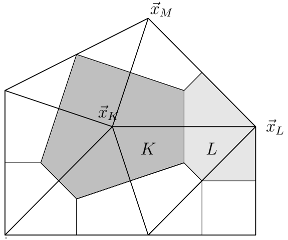


Quantum dots



Eymard R., Gallouët T., and Herbin R., *The Finite Volume Method*. In Ciarlet P.G. and Lions J.L., editors, *Handbook of Numerical Analysis*, Vol. VII, pages, 715–1022. North-Holland, 2000.

Voronoi Finite Volumes

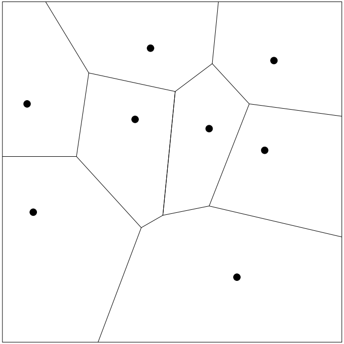


- L^∞ stability, local maximum principle
- Existence of discrete solution
- L^1 contraction, uniquness of the discrete solution
- Discrete $L^2(0, T; H^1(\Omega))$ estimate depending on $\operatorname{reg}\mathcal{D}$ and not on $\operatorname{size}\mathcal{D}$
- Space and time translate estimate not depending on $\mathcal{D}$
- Convergence to weak solution for $\operatorname{size}(\mathcal{D}) \to 0$ while $\operatorname{reg}(\mathcal{D}) \geq \rho$

Fuhrmann J., and Langmach H., *Stability and existence of solutions of time-implicit finite volume schemes for viscous nonlinear conservation laws*. App. Num. Math., **37**:201–230, 2001.

Eymard R., Fuhrmann J., and Gärtner K., *A finite volume scheme for nonlinear parabolic equations derived from 1D local Dirichlet problem*. Numerische Mathematic, **102**(3):463–495, 2006.

Given a set of points $S \subset \mathbb{R}^d$. For each $p \in S$, the Voronoi cell of p, $V(p)$, is:
$V(p) = \{x \in \mathbb{R}^d \mid \forall q \in S \; |x - p| \leq |x - q|\}$.



Georgy F. Voronoy (1868-1908)

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The Voronoi Diagram

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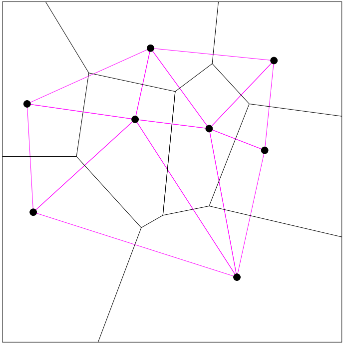

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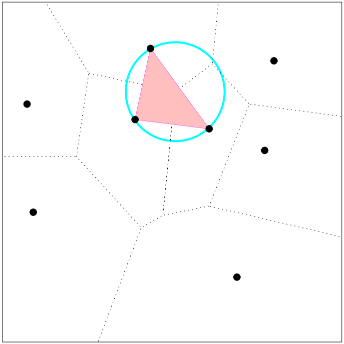

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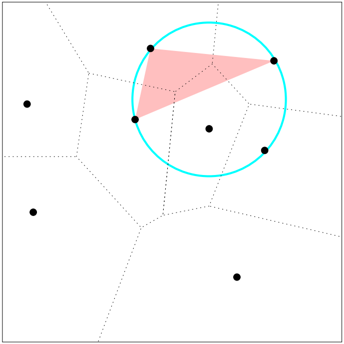

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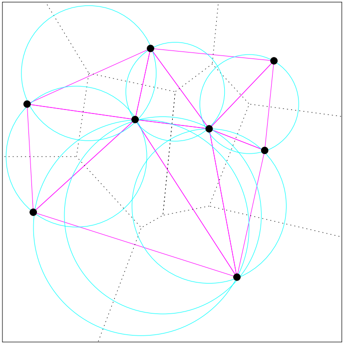

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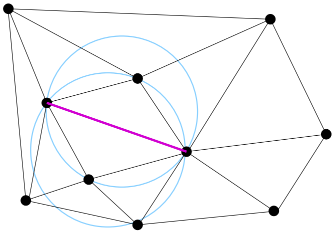
Delaunay Triangulation

Some properties of Delaunay triangulation. (Explained in 2D, generalized to higher dimensions.)

- A simplex is locally Delaunay if it has an empty circumcircle.
- edge flip - local transformation between Delaunay and non-Delaunay simplices.
- Incremental construction and updating.
- Generalize to 3 and higher dimensions.



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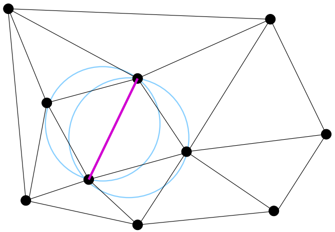

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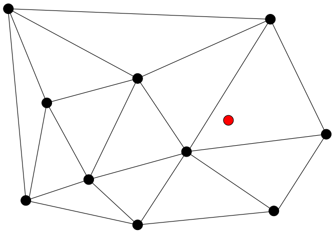

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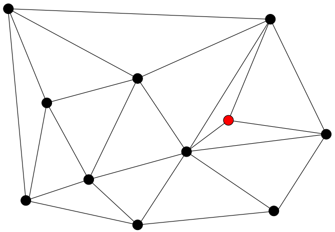

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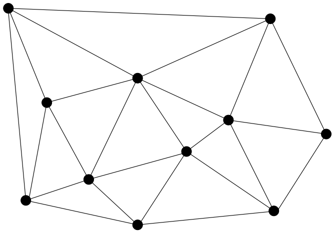

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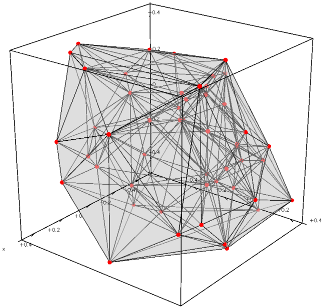
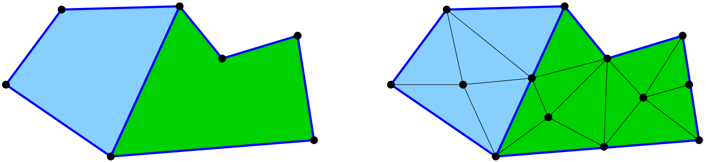


Boundary Conforming Delaunay Mesh

- Given any domain $\Omega \in \mathbb{R}^d$. The Delaunay mesh $\mathcal{T}$ is a partition of Ω by a set of Delaunay simplices and the boundary $\partial\Omega$ is represented by a union of simplices of $\mathcal{T}$.
- The dual Voronoi diagram of a Delaunay mesh may not conform to the input boundary.
- $\mathcal{T}$ is a boundary conforming Delaunay mesh of Ω if the diametric sphere of every boundary simplex of $\mathcal{T}$ is empty.



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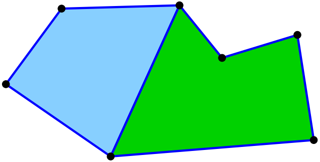

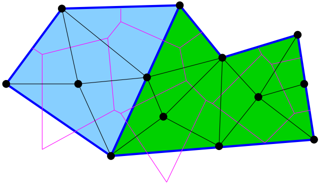

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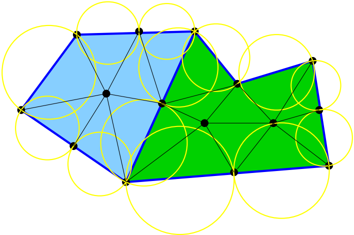

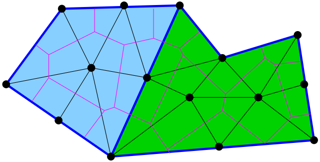

The Task

For a given 3D domain Ω, find a tetrahedral mesh $\mathcal{T}$, such that

1. $\mathcal{T}$ is a boundary conforming Delaunay mesh (conformity).
2. Tetrahedra of $\mathcal{T}$ are well-shaped (quality guarantee).
3. The number of tetrahedra of $\mathcal{T}$ is small (size guarantee).

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State-of-the-art:

- ▷ Most of the mesh generation methods can satisfy both *2* and *3*, but do not respect the conformity.
- ▷ Methods that theoretically guarantee the *1* have strong limitations.
- ▷ The big gap: lack of implementation.

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- The big gap: lack of implementation.

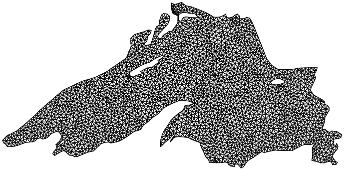
The Goals:

- Further the theoretical work for this problem.
- Implement robust and efficient program for various applications.

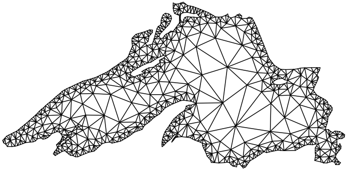
Outline

1. Introduction
2. **Delaunay Refinement**
3. Adaptive Refinement and Coarsening
4. Application Examples
5. Conclusion

Delaunay refinement – mesh refinement based on Delaunay triangulations.
The output is a boundary conforming Delaunay mesh.



$30° \leq \theta_{out}$, **uniform size** [Chew]



$20.7° \leq \theta_{out}$, **graded size** [Ruppert]
Implemented in *Triangle* [Shewchuk]

Chew P.L., *Guaranteed-quality triangular meshes*. Technical Report TR-89-983, Department of Computer Science, Cornell University, 1989.

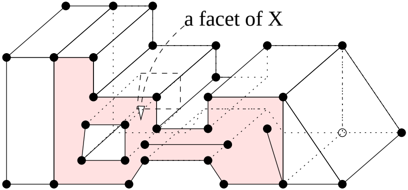
Ruppert J., *A Delaunay refinement algorithm for quality 2-dimensional mesh generation*. J. Algorithms, **18**(3):548–585, 1995.

Shewchuk J.R., *Delaunay refinement mesh generation*. PhD thesis, Department of Computer Science, Carnegie Mellon University, Pittsburgh, Pennsylvania, 1997.

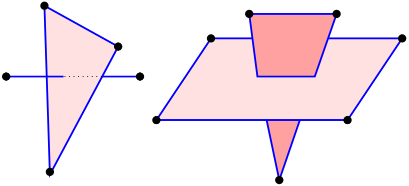
Piecewise Linear Complex

A piecewise linear complex (PLC) [Miller *et al.*'1996] is a set of polytopes X with the following properties:

1. The set X is closed under taking boundaries, i.e., for each $P \in X$ the boundary of P is a union of polytopes in X.
2. X is closed under intersection.
3. If $dim(P \cap Q) = dim(P)$ then $P \subseteq Q$, and $dim(P) < dim(Q)$.

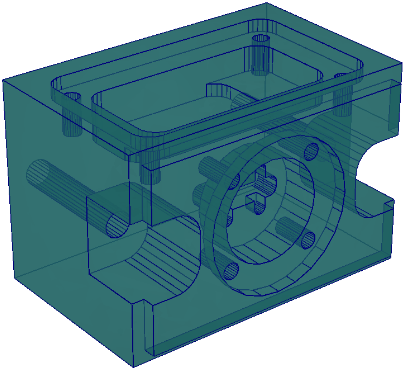
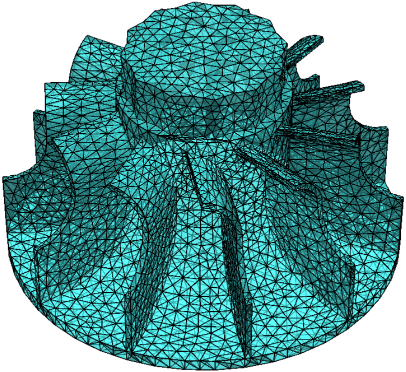


A PLC



non-PLCs

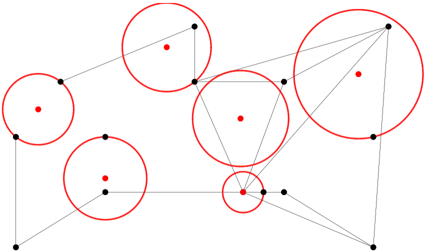
Piecewise Linear Complex

Local Feature Size

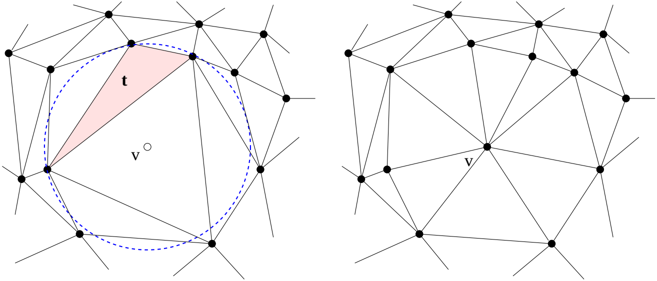
Given a PLC X, the local feature size [Ruppert'1995] at a point $p \in X$, lfs(p), is the radius of the smallest ball centered at p that intersects 2 non-incident boundaries of X.

- Bounded minimum, i.e., for any $p \in X$, lfs$(p) \geq$ lfs$_{min} > 0$.
- Lipschitz function, i.e, for $p, q \in X$, lfs$(p) -$ lfs$(q) \leq |p - q|$.



The Basic Idea

- Add the circumcenter of each badly-shaped element. Update the Delaunay triangulation with the new point.
- Prove termination: show that new edges at v never get too short, i.e., $|v - w| \geq \text{lfs}_{min}$.
- Prove well-graded: show that lfs(v) is bounded, i.e., for $D > 0$, lfs$(v) \leq D\ |v - w|$.



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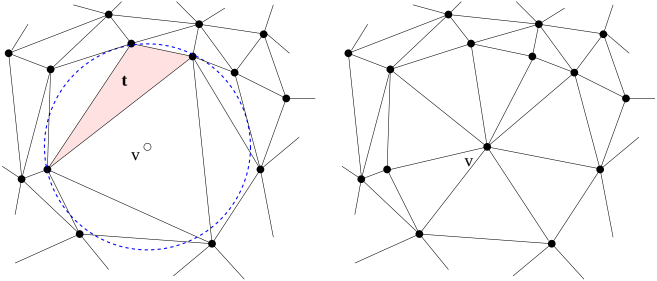

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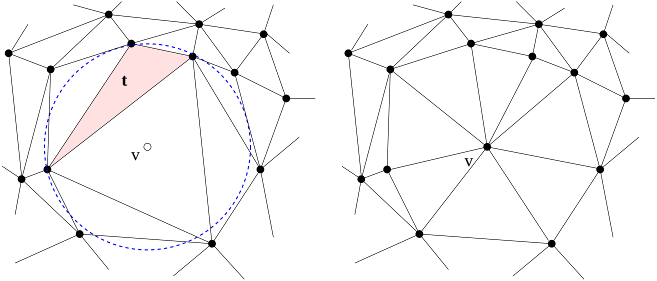
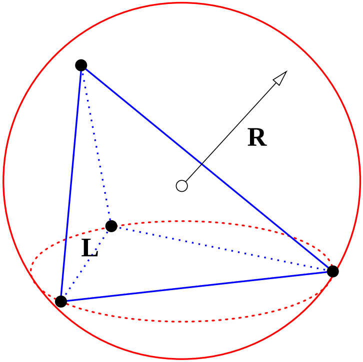


A Quality Measure for Tetrahedron

The radius-edge ratio of a tetrahedron t is the ratio between the radius R of its circumsphere and the length l of the shortest edge, i.e., $Q(t) = R/L$.



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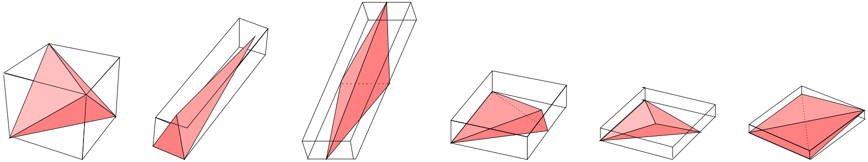
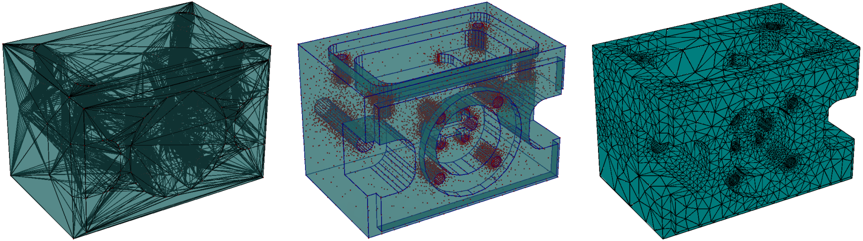
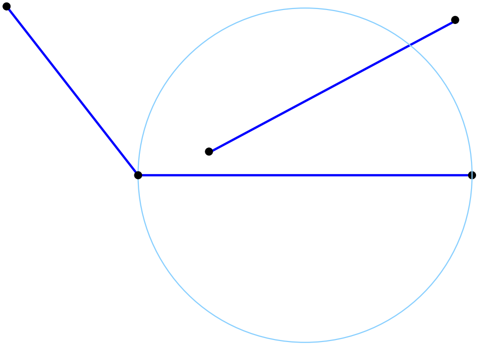


Regular	Needle	Spindle	Wedge	Cap	Sliver
0.612	4.1	26.27	6.74	3.57	0.707

Algorithm DelaunayRefine(X: PLC, ρ_0: radius-edge ratio bound)
Initialize a set $\mathcal{P}$ of vertices of X;
Initialize a Delaunay tetrahedralization, $\mathcal{D}(\mathcal{P})$;
repeat:
Find a new point v by the point generating rules;
Add v to $\mathcal{P}$, update $\mathcal{D}(\mathcal{P})$;
until {no new point can be inserted}.
return current $\mathcal{D}(\mathcal{P})$;



R1. If a subsegment is *encroached*, split it at its midpoint.



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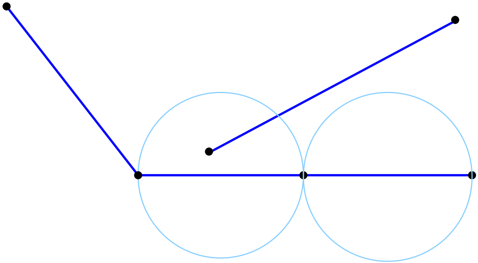

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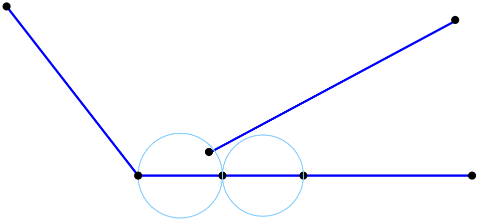

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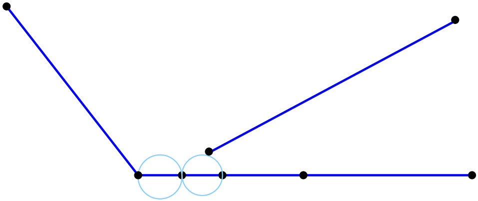

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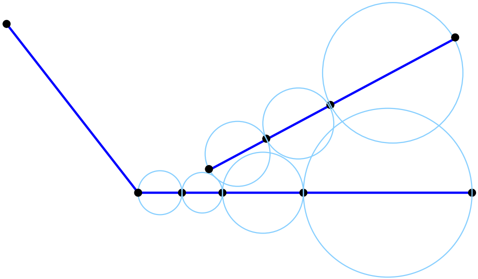
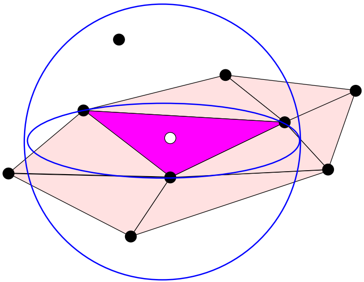


R2. If a subface f is encroached, try to insert its circumcenter c.
If c encroaches upon any subsegment, then reject c.
Instead, use **R1** to split all encroached subsegments.



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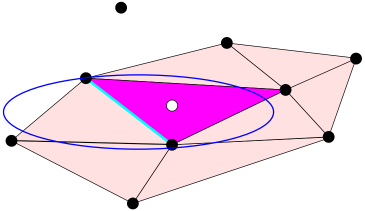

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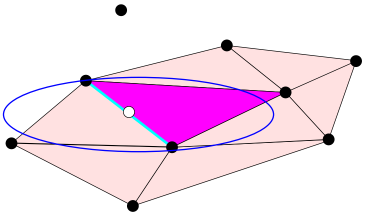
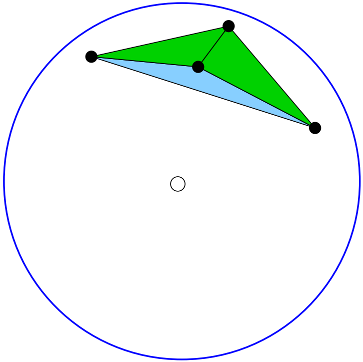


R3. If a tet t is bad ($Q(t) > \rho_0$), try to insert its circumcenter c.
If t encroaches upon any subsegment or subface, then reject c.
Instead, use **R1** and **R2** to split all encroached subsegments and subfaces.



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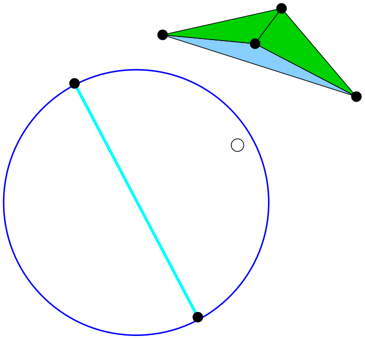

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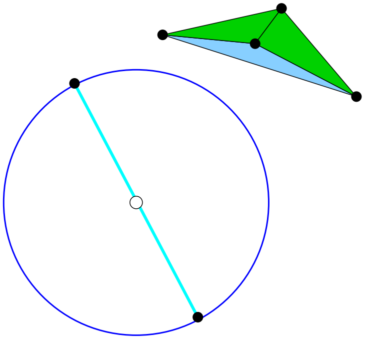
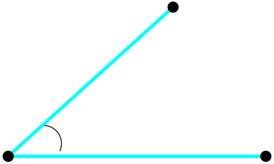
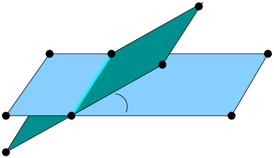


Given a PLC X, define two types of input angles of X.

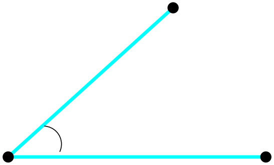


segment-segment angle

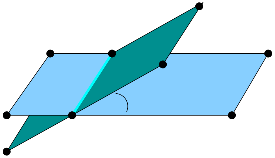


facet-facet (dihedral) angle

Given a PLC X, define two types of input angles of X.



segment-segment angle



facet-facet (dihedral) angle

The input angle condition:

(1) No segment-segment is less than $60°$.

(2) No facet-facet angle is less than $90°$.

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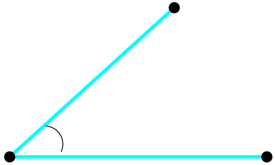
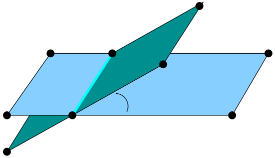


segment-segment angle



facet-facet (dihedral) angle

The input angle condition:

(1) No segment-segment is less than 60°.

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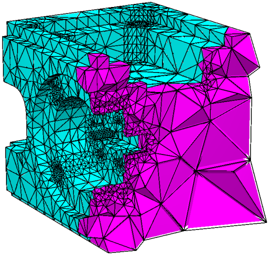
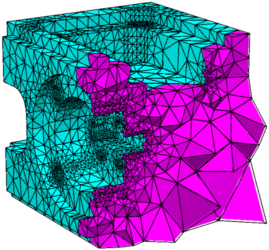
Bounded edge length. For any newly inserted vertex v, lfs$(v) \leq D\,|v - w|$, where $D = \frac{(3+\sqrt{2})\rho_0}{\rho_0 - 2}$.

Analysis [Shewchuk'1997]

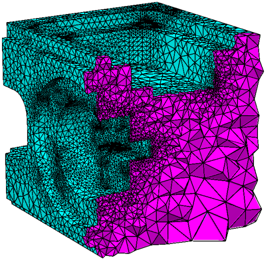
Termination. Assume that X satisfies the input angle condition. Then the algorithm terminates with a radius-edge ratio ρ_0, where $\rho_0 > 2$.



$Q(t) \geq 2.0$
$7,862$ nodes, 1.5 sec.

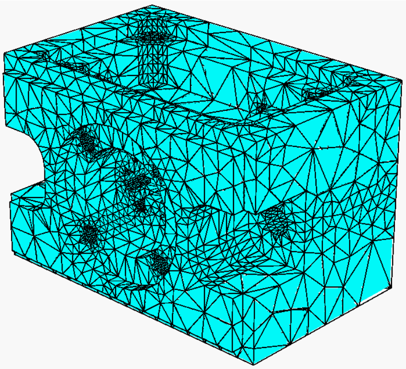


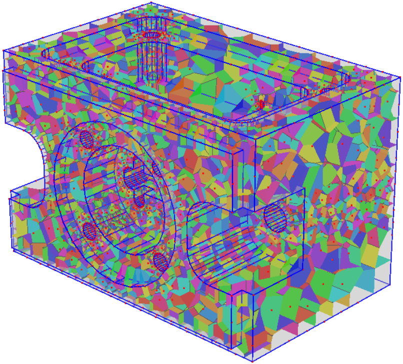
$Q(t) \geq \sqrt{2}$
$14,653$ nodes, 2.5 sec.



$Q(t) \geq 1.1$
$54,560$ nodes, 8.7 sec.

Conformity. The output is a boundary conforming Delaunay mesh.



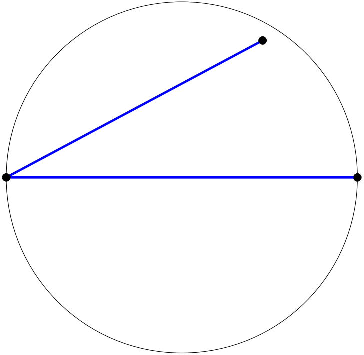


The problem of small angles

The input angle condition:

(1) No segment-segment is less than $60°$.

(2) No facet-facet angle is less than $90°$.



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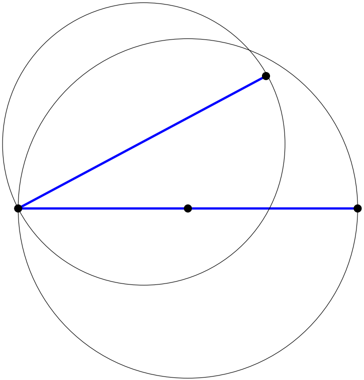

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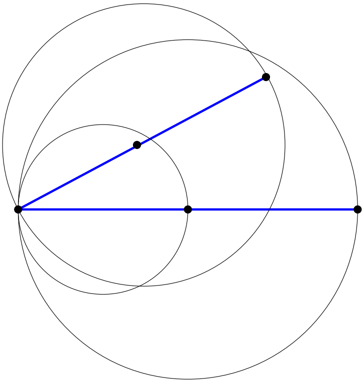

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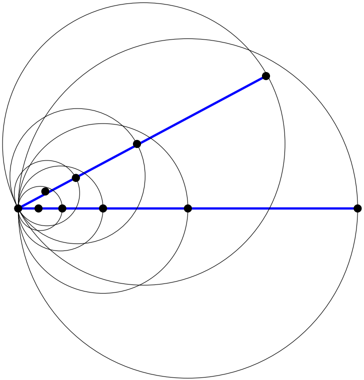

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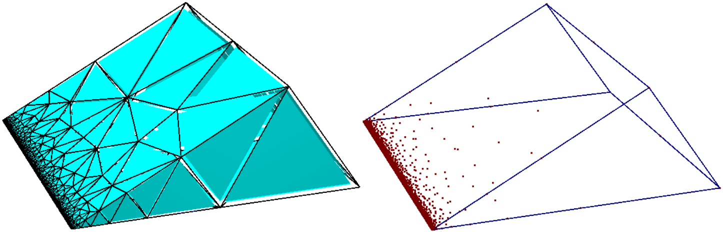
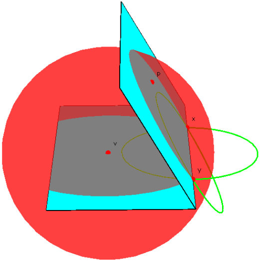


Improvement

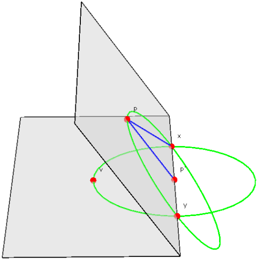
The relaxed input angle condition:

(1) No segment-segment angle is less than $60°$.

(2) No facet-facet angle is less than $\arccos \frac{1}{2\sqrt{2}} \approx 69.3°$.



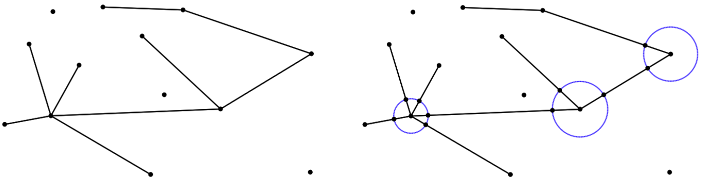
$|p - v| < |p - x|$



$|p - v| \geq \frac{1}{2\sqrt{2}\cos\theta}|p - x|$

Termination. Assume that X satisfies the relaxed input angle condition. Then the algorithm terminates with a radius-edge ratio ρ_0, where $\rho_0 \geq 2$.

The input segment condition: For any two segments S_1 and S_2 of X, if they are not collinear, then $|S_1| = |S_2|$.



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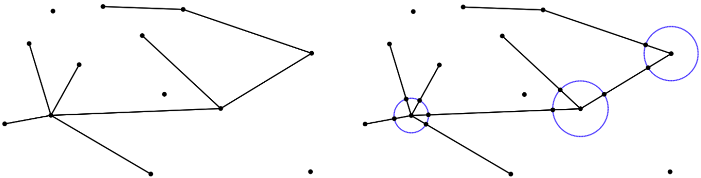

Improved Mesh Quality. Assume that X satisfies the relaxed input angle condition and the input segment condition. Then the algorithm terminates with a radius-edge ratio ρ_0, where $\rho_0 > \sqrt{2}$.

Improvement

The relaxed input angle condition:

(1) No segment-segment angle is less than 60°.

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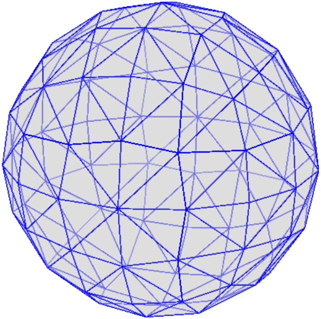

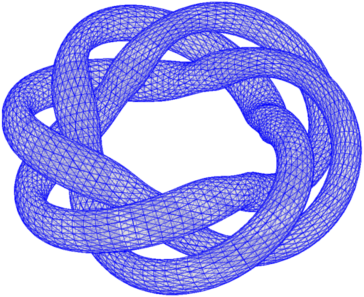

minimum segment-segment angle $> 38^\circ$ $> 21^\circ$

Improvement

The modified point generation rule.

$R1^*$ If a segment S is encroached. Let v be its midpoint. Insert v only If:

(1) S is not *sharp*, or

(2) S is *sharp* and the cause of splitting s is an existing mesh vertex.

The modified point generation rule.

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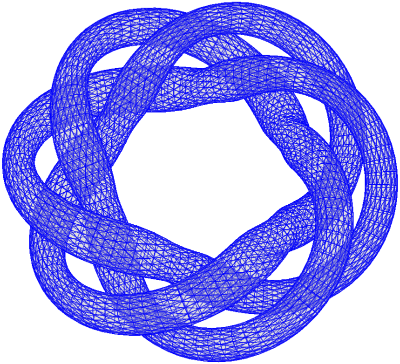
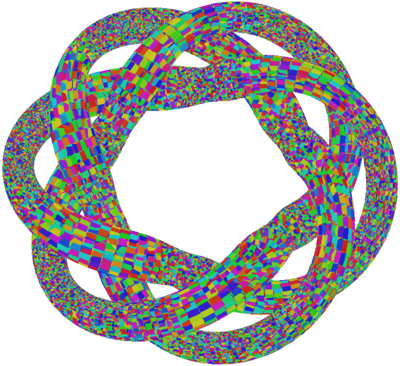
Termination. Assume that X has no facet-facet angle is less than $69.3°$, X satisfies the input segment condition, and $R1^*$ is used. Then the algorithm terminates with a radius-edge ratio ρ_0, where $\rho_0 > \sqrt{2}$.

Improvement

Termination. Assume that X has no facet-facet angle is less than $69.3°$, X satisfies the input segment condition, and $R1^*$ is used. Then the algorithm terminates with a radius-edge ratio ρ_0, where $\rho_0 > \sqrt{2}$.



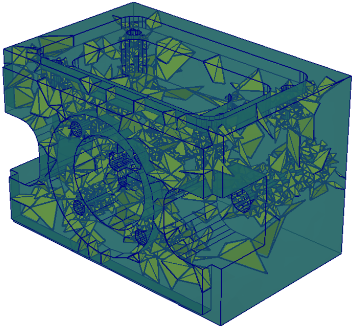
Surface mesh



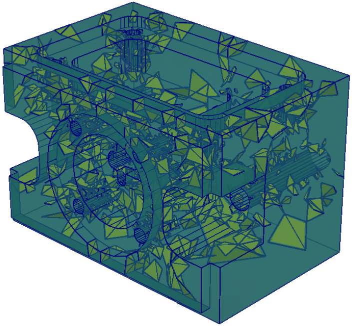
Boundary conforming Voronoi diagram

The problem of slivers

- Slivers ("round" and very "flat" tetrahedra) are not removed.
- Bound the largest (or smallest) dihedral angle, θ_{dihed} (open question).
- In practice, Delaunay refinement works very well by considering θ_{dihed} as an additional quality measure.



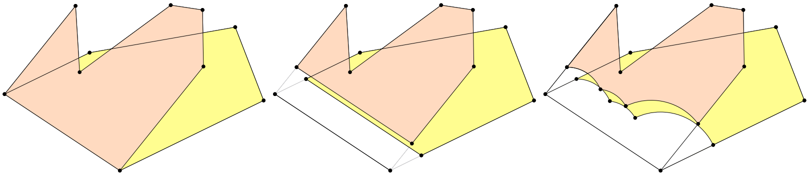
$Q(t) > 2$ ($\theta_{max} = 179.75°$)



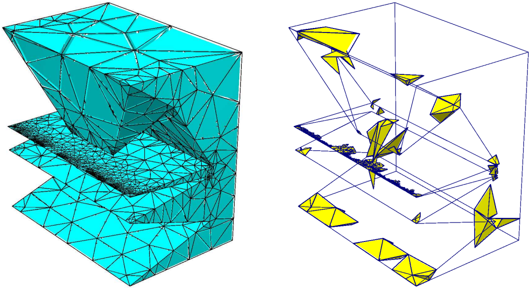
$Q(t) > \sqrt{2}$ ($\theta_{max} = 179.6°$)

Cheng & Poon'03, Cheng & Day'04,05, Pav & Walkington'04, ...

- ▷ Create protect regions to separate small input angles.
- ▷ Use Delaunay refinement to mesh the interior.



- Good mesh quality inside the mesh domain.
- Remaining bad quality tets are close to small input angles.
- No bound on time and space usage – not practical!
- No support of user-defined mesh sizing functions – not adaptive!



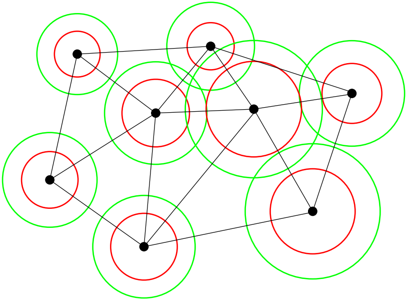
Quality tet mesh generated by *QualMesh* (T. Day).

Outline

1. Introduction
2. Delaunay Refinement
3. **Adaptive Refinement and Coarsening**
4. Application Examples
5. Conclusion

The Idea

- For each point p, assume there are two virtual balls, one protect ball (shown in red), and one sparse ball (shown in green).
- Generate candidates by the Delaunay refinement rules.
- Insert points if they are outside the neighboring protect balls.



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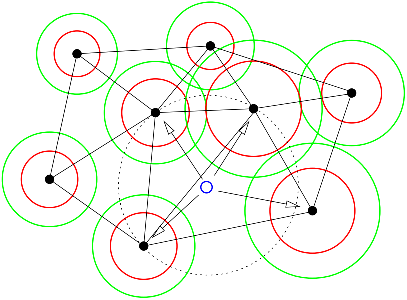

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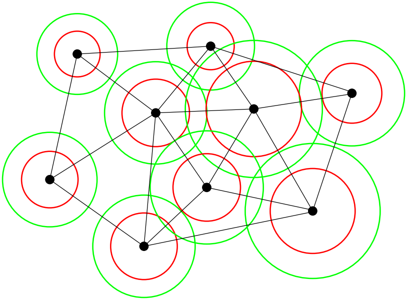
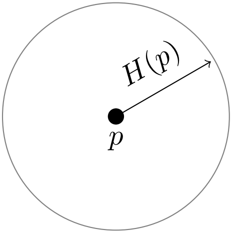


How to Decide the Radii?

- Use a sizing function, $H : \Omega \to \mathbb{R}^+$.
- Introduce two parameters: α_1 and α_2.
- The radii of the sparse and protect balls are $\alpha_1 H(p)$ and $\alpha_2 H(p)$, respectively.



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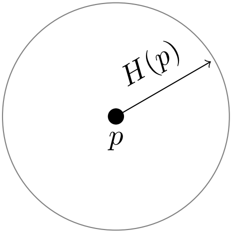

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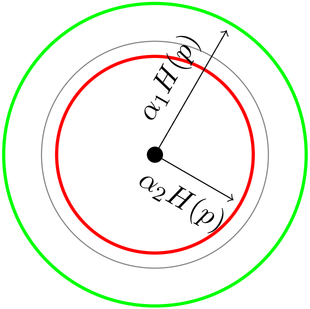


Outline of the algorithm

Algorithm: Adaptive Delaunay refinement.

Input: $\mathcal{T}$, ρ_0, H, α_1, α_2;

Repeat:

generate a new point v by the point generating rules[1];

If $|v - p| > \alpha_2 H(p), \forall p \in \mathcal{T}$ **then**

insert v and update $\mathcal{T}$;

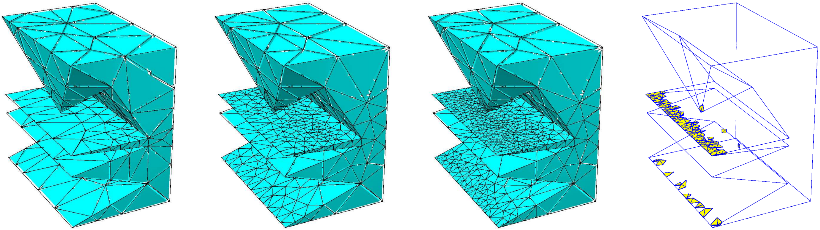
endif

Until no new point can be inserted;

1. $R3$ is modified: if $Q(t) > \rho_0$ or $|v - p| > \alpha_1 H(p), p \in \mathcal{T}$.

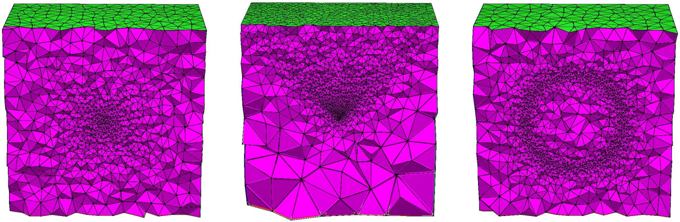
Analysis

- ▷ The algorithm terminates as long as $\alpha_2 > 0$ (No limitation on θ_{input}.)
- ▷ (Mesh quality) Most of the output tetrahedra have $Q(t) > \sqrt{2}$.
  The circumcenter of any bad quality tetrahedron is within distance $\sqrt{2}\alpha_2 H(p)$, where p is a point at sharp features.



$\alpha_2 = 0.5$ $\alpha_2 = 0.2$ $\alpha_2 = 0.1$ $\alpha_2 = 0.1$

Test of mesh conformity - $B = 2.0$, $\alpha_1 = \sqrt{2}$, $\alpha_2 = 0.05$

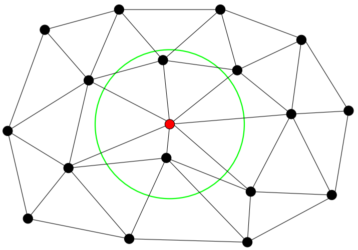


			S_v	L_v	S_v	L_v	S_v	L_v
	$<$	0.5	0	0	0	0	0	0
0.5	$-$	$1/\sqrt{2}$	58	0	0	0	0	0
$1/\sqrt{2}$	$-$	1	3221	1	283	0	0	0
1	$-$	$\sqrt{2}$	15062	113	10778	14	1927	49
$\sqrt{2}$	$-$	2	4246	3867	1187	1044	94186	12594
2	$-$	$2\sqrt{2}$	0	18606	0	11190	12276	95746
	$>$	$2\sqrt{2}$	0	0	0	0	0	0

$S_V = S(v)/H(v)$, $L_v = L(v)/H(v)$, where $S(v)$ and $L(v)$ denote the lengths of the shortest edge and longest edge among all edges connecting at v.

Coarsening

- For each point p, if there exist a point q which is inside the sparse ball of p, e.g., $|p - q| < \alpha_1 H(p)$, then remove it.



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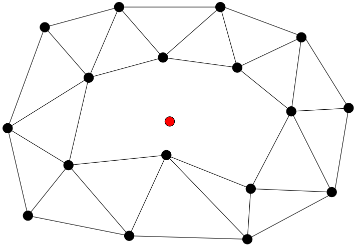
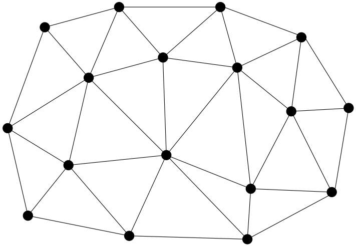


Coarsening

- For each point p, if there exist a point q which is inside the sparse ball of p, e.g., $|p - q| < \alpha_1 H(p)$, then remove it.



Algorithm: Adaptive Delaunay coarsening and refinement.

Input: $\mathcal{T}$, ρ_0, H, α_1, α_2;

---

for each $v \in \mathcal{T}$, **do**

  if $|v - p| < \alpha_1 H(v), p \in \mathcal{T}$, **then** remove v;

endfor

---

repeat:

  generate a new point v by the point generating rules[1];

  if $|v - p| > \alpha_2 H(p), p \in \mathcal{T}$ **then**

    insert v and update $\mathcal{T}$;

  endif

until no new point can be inserted;

1. $R3$ is modified: if $Q(t) > \rho_0$ or $|v - p| > \alpha_1 H(p), p \in \mathcal{T}$.

Outline

1. Introduction
2. Delaunay Refinement
3. Adaptive Refinement and Coarsening
4. **Application Examples**
5. Conclusion

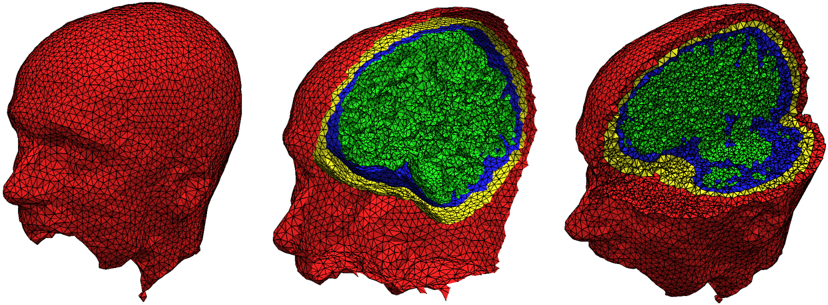
TetGen – A Delaunay Tetrahedral Mesh Generator.

- H can be automatically estimated based on the input geometric data; alternatively, H can be supplied by the user.
- Parameters ρ_0, α_1, and α_2 can be adjusted at the run time.
- Remove slivers by (1) using a dihedral angle bound (adjustable), and (2) mesh optimization and smoothing.
- Capable of dealing with arbitrary 3D PLCs.
- Robust algorithms and implementation.
- Memory efficient.

Freely available for academic and research use (http://tetgen.berlios.de) .

EEG/MEG-source localization

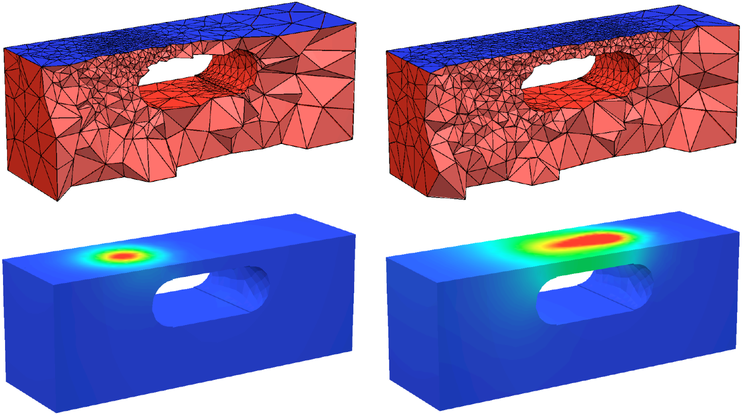
A PLC model of human brain which consists of four surface meshes: skin (red), outer and inner skull (yellow and blue), and cortex (green). (Institut für Biomagnetismus und Biosignalanalyse, Uni. Münster).



Input: $20,301$ points and $40,638$ triangles. Output: $85,312$ points, $528,727$ tetrahedra. CPU time: 18 sec.

Surface Hardening Simulation

Adaptive boundary conforming Delaunay mesh refinement and coarsening applied in the program WIAS Sharp.



Simulation of transient heat conduction (at different times).

Outline

1. Introduction
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Conclusion

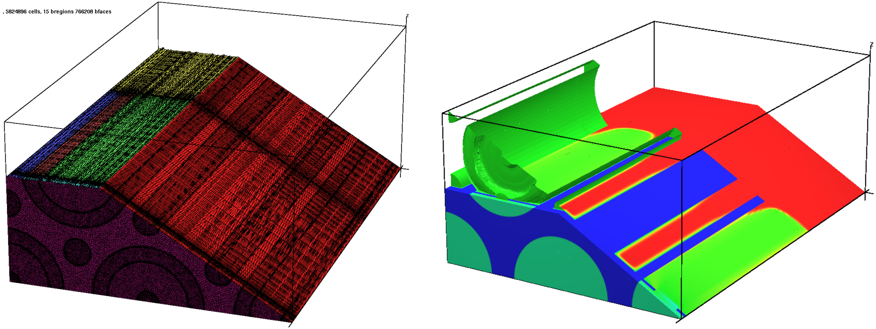
- Boundary conforming Delaunay meshes are well-suited for solving non-linear convection-diffusion problem by finite volume method.
- In 3D, many theoretical questions for creating such mesh are still open. Big gap remains between theory and practice.
- The Delaunay refinement algorithm has been extended:
  1. mesh quality and mesh size remain provable;
  2. no limitation on the input angle;
  3. adaptive refinement and coarsening.
- The **TetGen** program which implements fast, robust, quality-guaranteed algorithms has been used in applications.

Main References

- ▷ Eymard R, Gallouët T, and Herbin R, *Finite Volume Methods*. Handbook of Numerical Analysis, Vol. VII, Elsevier Science B. V., 2000.
- ▷ Shewchuk J, *Tetrahedral Mesh Generation by Delaunay Refinement*. In Proc. 14th Annu. Sympos. Comput. Geom., 1998.
- ▷ Si H, Gärtner K, *Meshing Piecewise Linear Complex by Constrained Delaunay Tetrahedralizations*. In Proc. 14th International Meshing Roundtable, San diego, CA, 2005.
- ▷ Si H, *Adaptive Tetrahedral Mesh Generation by Constrained Delaunay Refinement*. WIAS Preprint 1176, 2006.
- ▷ TetGen, A Tetrahedral Mesh Generator and 3D Delaunay Triangulator. *http://tetgen.berlios.de*

Future Work

- ▷ Sliver removal.
- ▷ Boundary conforming Delaunay for $\theta_{input} < 69.3^\circ$.
- ▷ Anisotropic boundary conforming Delaunay mesh generation.



An anisotropic mesh (left) and the numerical solution (right).