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**Institutskolloquium**

# Principle of Linearized Stability and Invariant Manifold Theorem for Semilinear Hyperbolic Systems with Applications to Semiconductor Laser Dynamics

## Overview

- ▷ Introduction
  - ▷ Principle of linearized stability
  - ▷ Growth and spectral bound, exponential dichotomy for linearized dynamical systems
  - ▷ Center manifold theorem
- ▷ Linearization of semilinear hyperbolic systems
  - ▷ The variation of constants formula
  - ▷ Sun star calculus
- ▷ Stability and dichotomy for linearized hyperbolic systems
  - ▷ Proof of principle of linearized stability and center manifold theorem for semilinear hyperbolic systems
- ▷ Applications

## Dynamical System

Given a dynamical system

$$\frac{d}{dt}x = f(x).$$

- State $x \in X$
- ODE:
  - $X = \mathbb{R}^n$
  - $f : \mathbb{R}^n \to \mathbb{R}^n$ ist $C^k$ smooth
- PDE:
  - $X$ Banach-space
  - $f$ a densely defined operator

## Determining the stability of stationary states

Let $x_0$ be a stationary state.

1. Linearize in $x_0$:
$$\frac{d}{dt}h = Df(x_0)h.$$
2. Determine the stability of the linearized problem:
   - Locate the spectrum of $Df(x_0)$.
3. Prove that the nonlinear problem is stable near $x_0$.

**Theorem (Principle of linearized stability)**

*Suppose there exists $s < 0$, so that for all $\lambda \in \sigma(Df(x_0))$*

$$\mathfrak{Re}\lambda \leq s < 0.$$

*Then $x_0$ is exponentially stable.*

## Approximation of the nonlinear dynamics via the linearized dynamics

- For the proof we need that the linearization $Df(x_0)$ is a good approximation for $f$ near $x_0$.

- PDE: The operator $f$ contains nonlinear Nemytskij operators. Their differentiability properties depend on the topology of the Banach-space $X$.
  - Usually it is not enough to consider only one Banach space $X$. Often we need a triple or even scale of Banach spaces.

## Stability of the linearized problem

- As is well known in finite dimensions the stability of the linear system $\frac{d}{dt}h = Df(x_0)h$ is determined by the eigenvalues (spectrum) of the matrix $Df(x_0)$.

- In infinite dimensions, where $X$ is a Banach-space, the issue is more complex.

- The appropriate abstract setting is provided by the theory of $C_0$ semigroups $\left(e^{At}\right)_{t\geq 0}$ of bounded linear operators on the Banach-space $X$.

## Growth and spectral bound

### Definition

Let $A = Df(x_0)$ be a generator of a $C_0$ semigroup $e^{At}$. The spectral bound $s(A)$ ist defined as

$$s(A) := \sup \{\mathfrak{Re}\, z \mid z \in \sigma(A)\} .$$

The growth bound $\omega(A)$ is per definitionem

$$\omega(A) := \inf \left\{\omega \in \mathbb{R} \mid \exists_{M=M(\omega)>0} : \left\|e^{At}\right\| \leq M e^{\omega t} \text{ for } t \geq 0\right\} .$$

- ▷ $\omega(A) = s(A)$ for ODEs, DDEs, semilinear parabolic PDEs.
- ▷ In general: $\omega(A) \geq s(A)$, equality must not hold.
- ▷ Warning: There exists a counterexample of a $2d$ wave equation with

$$\omega(A) > s(A).$$

## Determining the growth bound $\omega(A)$

### Proposition

*For $t > 0$*

$$\omega(A) = \frac{\log r(e^{At})}{t},$$

*where $r(e^{At}) := \sup\left\{|z| \mid z \in \sigma(e^{At})\right\}$ denotes the spectral radius of the semigroup $e^{At}$.*

Method for determining the growth bound $\omega$:

- Calculate $\sigma(A)$ by solving spectral problem.
- Important open question for hyperbolic PDEs: Can the unknown spectrum $\sigma(e^{At})$ of the semigroup be calculated from the spectrum $\sigma(A)$ of the generator $A$ (the equations of the PDE) ?

### Theorem

*For hyperbolic systems in $1d$ the answer is positive: $\omega(A) = s(A)$.*

## Existence of center manifolds

Assumptions:

- $s(A) \leq 0$
- $E_c := \sigma(A) \cap i\mathbb{R} \neq \emptyset$
- Spectral gap: There exists $\delta > 0$ such that

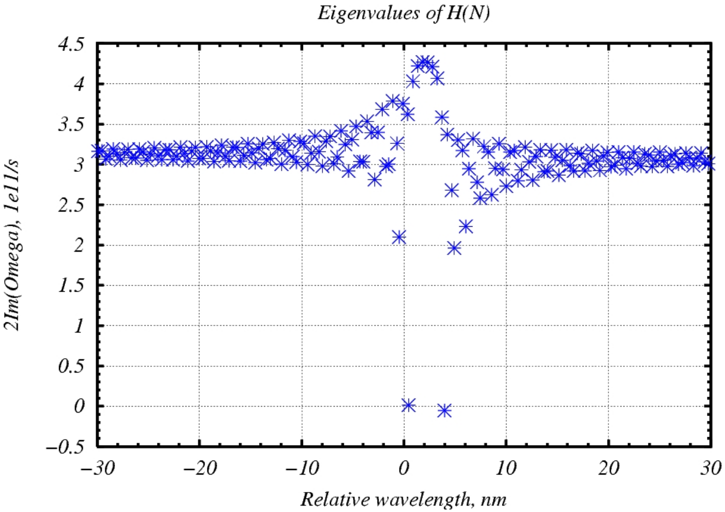
$$\{z \in \mathbb{C} \mid -\delta < \mathfrak{Re}\, z < 0\} \subset \rho(A).$$

Let $\pi_c : X^{\mathbb{C}} \to X^{\mathbb{C}}$ denote spectral projection corresponding to the critical eigenvalues $E_c$, where $X^{\mathbb{C}}$ denotes complexification of $X$. Further let

$$X_c := X \cap \mathrm{Im}(\pi_c) = X \cap \bigoplus_{\lambda \in E_c} \bigcup_{j=1}^{\infty} \mathrm{Ker}\,(\lambda \mathrm{Id} - A)^j\,,$$

$$X_s := X \cap \mathrm{Ker}(\pi_c).$$

## Spectrum of the traveling wave operator (LDSL)



## Spectral gap and exponential dichotomy

- For ordinary differential equations, delay equations and semilinear parabolic PDEs it is known, that the spectral gap condition generates an exponential dichotomy on the spectral decompositions.
  - Let $T_c(t) := e^{At}_{|X_c}$ and $T_s(t) := e^{At}_{|X_s}$
  - $\exists c > 0 : \|T_s(t)\| \leq ce^{-\delta t} \quad$ for $t \geq 0$
  - $\forall \epsilon > 0 \exists d > 0 : \|T_c(-t)\| \leq de^{\epsilon t}$
- Exponential dichotomy is necessary for the proof of center manifold theorem.
  - If there is no exponential dichotomy, it is known due to a result of Mane, that the critical eigenspace $X_c$ does not persist under small nonlinear perturbations.

## Center manifold theorem

### Theorem

*There exists a neighbourhood $U$ of zero in $X$ and a smooth graph $\gamma : X_c \cap U \to X_s$ with the following properties:*

- *the manifold $M := \{x_0 + x_c + \gamma(x_c) \mid x_c \in U \cap X_c\}$ is locally invariant and exponentially attractive with respect to the nonlinear semiflow,*
- *any solution $u : \mathbb{R} \to x_0 + U$ lies on $M$,*
- *the trajectories on $M$ are governed by the equation*

$$\frac{d}{dt}x_c = Df(x_0)x_c + \pi_c r(x_c + \gamma(x_c)),$$

*where the remainder $r$ is of order $2$.*

## Core problems for existence of invariant manifolds in infinite dimensional dynamical systems

- Does a spectral gap generate an exponential dichotomy ?
- Does the evolution equation form a smooth semiflow on $X$ ? Is the solution map linearizable with respect to the norm of $X$ ?
- If yes, for which Banach-spaces are both properties fulfilled ?

These issues have been resolved for large classes of semilinear parabolic PDEs and DDEs, but not for hyperbolic PDEs.

## A general class of semilinear hyperbolic systems

$$
\text{(SH)} \begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + H(x, u(t,x), v(t,x)), \\ v(t,l) = D\,u(t,l), \\ u(t,0) = E\,v(t,0). \end{cases}
$$

- $x \in \,]0,l[$, $t > 0$
- $u \in \mathbb{R}^{n_1}$, $v \in \mathbb{R}^{n_2}$, $n = n_1 + n_2$, $D$, $E$ matrices
- $K(x) = \mathrm{diag}\,(k_j(x))_{1\le j\le n}$, $k_j \in C^1([0,l],\mathbb{R})$,

$$
k_j < 0 \quad 1 \le j \le n_1, \qquad k_j > 0 \quad n_1 + 1 \le j \le n.
$$

- $H : \,]0,l[\, \times \mathbb{R}^n \to \mathbb{R}^n$ smooth in $(u,v)$, measurable in $x$

## Variation of constants formula

Let $T(t)$ be the reflection / shift semigroup generated by

$$\begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix}, \\ u(t,0) = E\,v(t,0), \quad v(t,l) = Du(t,l) \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x). \end{cases}$$

The nonlinearity $H : ]0,l[ \times \mathbb{R}^n \to \mathbb{R}^n$ generates a Nemytskij operator:
For $u : ]0,l[ \to \mathbb{R}^{n_1}$, $v : ]0,l[ \to \mathbb{R}^{n_2}$

$$\mathfrak{H}\,(u,v)\,(x) := H(x, u(x), v(x)).$$

Formally the variation of constants formula for $(\mathrm{SH})$ reads

$$\text{“}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T(t-s)\mathfrak{H}(u(s), v(s))\,ds.\text{”}$$

## Which choice of space $X$ ?

$$\text{``}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T(t-s)\mathfrak{H}(u(s), v(s))\, ds.\text{''}$$

It is tempting to take the Hilbert space $L^2(]0,l[\,,\mathbb{R}^n)$ for $X$:

- $T(t)$ is strongly continuous on $L^2$.
- The Nemytskij operator $\mathfrak{H}$ ist not well defined on $L^2$.
  - Need to truncate the nonlinearity $H$ so that the Nemytskij operator becomes well defined and globally Lipschitz on $L^2$.
- But still it is not Fréchet differentiable due to the (rather surprising) fact that $\mathfrak{H} : L^2 \to L^2$ is differentiable at some $(u,v) \in L^2(]0,l[\,,\mathbb{R}^n)$ if and only if for almost all $x \in\, ]0,l[$ the function $z \mapsto H(x,z)$ is affine.

## A good choice for $X$

$$“\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T(t-s)\mathfrak{H}(u(s),v(s))\,ds.”$$

Take

$$X := \Big\{ (u,v) \in C([0,l],\mathbb{R}^n) \mid u(0) = Ev(0), \quad v(l) = Du(l) \Big\}.$$

- $T(t)$ is strongly continuous on $X$.
- But $\mathfrak{H}$ maps $X$ out to a larger space: If $(u,v) \in X$ then $\mathfrak{H}(u,v) \notin X$ for almost any choice of $H$ and $(u,v)$.
- Need to enlarge the space $X$ !

## Enlarging $X$, the sun star space

Idea: Construct a larger space in terms of a combination of properties of the space $X$ and the semigroup $T$.

- Let $T^*(t) : X^* \to X^*$ be the adjoint semigroup
- Then $t \to T^*(t)x^*$ is not necessarily continuous (even not Bochner measurable, but weak star continuous). Let

$$X^{\odot} := \left\{ x^* \in X^* \mid \lim_{t \downarrow 0} \|T^*(t)x^* - x^*\| = 0 \right\}$$

  be the subspace on which $T^*$ is strongly continuous.
- Define $j : X \to X^{\odot *}$, $\langle jx, x^{\odot} \rangle := \langle x^{\odot}, x \rangle$ ($X^{\odot *} := (X^{\odot})^*$)
- $j$ is injective since $X^{\odot}$ is weak star dense in $X^*$, hence

$$X \overset{j}{\hookrightarrow} X^{\odot *}.$$

## Enlarging $X$, the sun star space

Put

$$T^{\odot}(t) := T^{*}(t)_{|X^{\odot}}.$$

By definition $T^{\odot}(t) : X^{\odot} \to X^{\odot}$ is a strongly continuous semigroup.
Again consider the adjoint semigroup $T^{\odot *}(t) = \left(T^{\odot}(t)\right)^{*} : X^{\odot *} \to X^{\odot *}$.

$$\begin{aligned} \forall x^{\odot} \in X^{\odot} : \langle T^{\odot *}(t) jx, x^{\odot}\rangle &= \langle jx, T^{\odot}(t) x^{\odot}\rangle \\ &= \langle T^{\odot}(t) x^{\odot}, x\rangle \\ &= \langle x^{\odot}, T(t) x\rangle \\ &= \langle j(T(t)x), x^{\odot}\rangle. \end{aligned}$$

- Hence $j(T(t)x) = T^{\odot *}(t) jx$ or

$$j \circ T(t) = T^{\odot *}(t) \circ j.$$

## Enlarging $X$, the sun star space

$$\begin{array}{ccc} T(t):X & \longrightarrow & T^*(t):X^* \\ & & \downarrow \\ T^{\odot *}(t):X^{\odot *} & \longleftarrow & T^{\odot}(t):X^{\odot} \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{T(t)} & X \\ j\downarrow & & j\downarrow \\ X^{\odot *} & \xrightarrow{T^{\odot *}(t)} & X^{\odot *} \end{array}$$

## The sun star space for hyperbolic systems with reflection boundary conditions

### Theorem

*$X^{\odot *}$ is isomorphic to $L^\infty([0,l],\mathbb{R}^n)$.*

$$\text{For } \begin{pmatrix} u \\ v \end{pmatrix} \in X : \quad T^{\odot *}(t)\begin{pmatrix} u \\ v \end{pmatrix} = T(t)\begin{pmatrix} u \\ v \end{pmatrix}.$$

The main advantages of using the space $X$ together with its sun dual $X^{\odot *}$ are based on the following two Lemmas:

### Lemma

*If $H(x,z)$ is measurable with respect to $x$ and smooth with respect to $z$ then the Nemytskij operator $\mathfrak{H}(u,v)(x) := H(x,u(x),v(x))$ is a smooth map from $X$ into $X^{\odot *}$.*

## Variation of constants formula

Moreover we get back from $X^{\odot *}$ into the small space $X$:

### Lemma

*Let $f : [0,T] \to X^{\odot *}$ be norm continuous. Then the weak-star integral*

$$t \mapsto \int_0^t T^{\odot *}(t-s) f(s)\, ds$$

*is norm continuous and takes values in $X$.*

### Definition (Variation of constants formula)

$(u,v) \in C([0,T],X)$ is called a mild (or weak) solution to $(\mathrm{SH})$ if

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = T(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \int_0^t T^{\odot *}(t-s) \mathfrak{H}(u(s), v(s))\, ds.$$

## Some straightforward consequences

### Theorem (Unique local existence)

*For any* $(u_0, v_0) \in X$ *there exists a* $\delta > 0$*, depending only on* $\|(u_0, v_0)\|_X$*,* $\mathfrak{H}$ *and* $T(t)$*, such that* $(\mathrm{SH})$ *has a unique mild solution* $(u, v) \in C([0, \delta], X)$ *with* $u(0) = u_0$*,* $v(0) = v_0$*.*

### Theorem

*Let* $z \in C([0, T], X)$ *be a weak solution of* $(\mathrm{SH})$*. Then there exists a neighborhood* $U$ *of* $z(0)$ *in* $X$ *such that for all* $y_0 \in U$ *there is a weak solution* $y \in C([0, T], X)$ *of* $(\mathrm{SH})$ *satisfying* $y(0) = y_0$*.*
*There exists a constant* $c > 0$ *such that for all* $y_0 \in U$

$$\|z(t) - y(t)\|_X \leq c\|z(0) - y_0\|_X.$$

## The smooth semiflow

Suppose there exists a weak solution $z \in C([0,T],X)$ of $(\mathrm{SH})$. Then according to the last Theorem there exists an open neighborhood $U$ of $z(0)$ in $X$ so that we can define a solution map

$$S^t : U \to X, \quad S^t(y_0) := y(t) \quad (t \in [0,T]).$$

### Theorem (Smooth semiflow property)

*For each $t \in [0,T]$ the map $S^t : U \to X$ is $C^k$ smooth. The map $(t,u) \mapsto S^t u$ is continuous from $[0,T] \times U$ into $X$. The total derivative $DS^t$, $\begin{pmatrix} \tilde{h_u}(t) \\ \tilde{h_v}(t) \end{pmatrix} = DS^t \begin{pmatrix} h_u \\ h_v \end{pmatrix}$ satisfies the equation*

$$\begin{pmatrix} \tilde{h_u}(t) \\ \tilde{h_v}(t) \end{pmatrix} = T(t) \begin{pmatrix} h_u \\ h_v \end{pmatrix} + \int_0^t T^{\odot *}(t-s) D\mathfrak{H}(u(s),v(s)) \begin{pmatrix} \tilde{h_u}(s) \\ \tilde{h_v}(s) \end{pmatrix} ds.$$

## Linearization of (SH)

Let $(u_0, v_0)$ be a stationary state. Then the last theorem states that the linearized flow $DS^t(u_0, v_0)$ is given by the mild solutions to the linearized system

$$
\text{(LH)} \begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} \\ \qquad\qquad\qquad\qquad + \partial_{(u,v)} H(x, u_0(x), v_0(x)) \begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix}, \\ v(t,l) = D u(t,l), \quad u(t,0) = E\, v(t,0). \end{cases}
$$

### Proposition

*The linearized flow $DS^t(u_0, v_0)$ is a $C_0$ semigroup $e^{At}$ on $X$ with infinitesimal generator*

$$
A\begin{pmatrix} u \\ v \end{pmatrix} = K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} + \partial_{(u,v)} H(x, u_0(x), v_0(x)) \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}.
$$

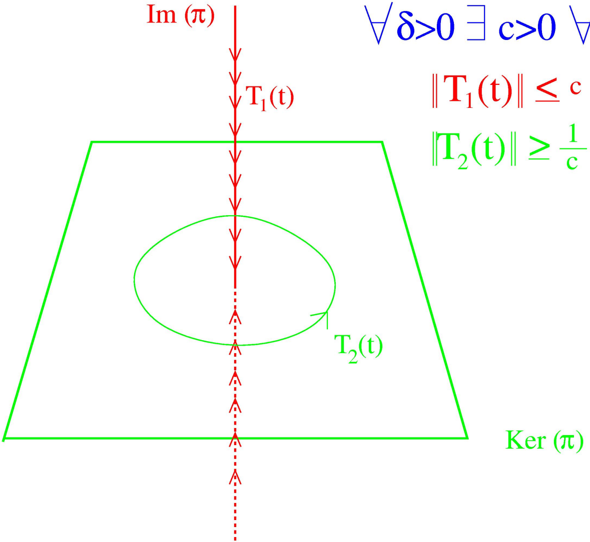
## $(\alpha, \beta)$ exponential dichotomy

### Definition

Let $\alpha < \beta$. $A$ has a $(\alpha, \beta)$ exponential dichotomy, if there exists a projection $\pi : X^{\mathbb{C}} \to X^{\mathbb{C}}$ such that

- $\pi e^{At} = e^{At}\pi$
- For $T_1(t) := e^{At}_{|\mathrm{Im}\,(\pi)}$ and $T_2(t) := e^{At}_{|\mathrm{Ker}(\pi)}$
  - $\omega(T_1(t)) \leq \alpha$
  - $T_2(t)$ extends to a group with $\omega(T_2(-t)) \leq -\beta$.

## $(\alpha, \beta)$ exponential dichotomy




$$\forall \delta > 0 \; \exists \; c > 0 \; \forall \; t \geq 0$$

$$\|T_1(t)\| \leq c\, e^{(\alpha+\delta)t}$$

$$\|T_2(t)\| \geq \frac{1}{c}\, e^{(\beta-\delta)t}$$

## Characterization of exponential dichotomy

### Theorem

*The following are equivalent:*

- $A$ *has a* $(\alpha, \beta)$ *exponential dichotomy.*
- $\forall_{t>0}: \quad \{\lambda \in \mathbb{C} \mid e^{\alpha t} < |\lambda| < e^{\beta t}\} \subset \rho(e^{At}).$
- $\exists t_0 > 0: \quad \{\lambda \in \mathbb{C} \mid e^{\alpha t_0} < |\lambda| < e^{\beta t_0}\} \subset \rho(e^{At_0}).$

- Exponential dichotomy means that there is a circular spectral gap for the semigroup $e^{At}$.
- Does a spectral gap condition on $A$ imply the presence of a circular spectral gap for $e^{At}$ ?

## Spectral mapping theorems for linearized hyperbolic systems

### Theorem

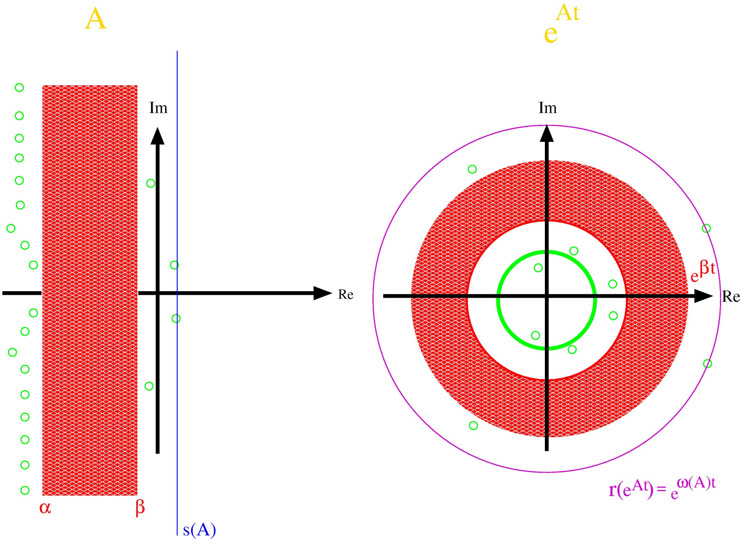
- $\sigma(e^{At}) \setminus \{0\} = \overline{e^{\sigma(A)t}} \setminus \{0\}$ *in* $L^2([0,l],\mathbb{C}^n)$
- *In* $X^{\mathbb{C}} = \{(u,v) \in C([0,l],\mathbb{C}^n) \mid u(0) = Ev(0),\ v(l) = Du(l)\}$ *for all* $\alpha < \beta$ *and* $t > 0$ *we have*

$$\{z \in \mathbb{C} \mid \alpha < \mathfrak{Re}\, z < \beta\} \subset \rho(A)$$
$$\Leftrightarrow \left\{z \in \mathbb{C} \mid e^{\alpha t} < |z| < e^{\beta t}\right\} \subset \rho(e^{At}).$$

### Corollary

*If* $\alpha < \beta$ *and* $\{\lambda \in \mathbb{C} \mid \alpha < \mathfrak{Re}\, \lambda < \beta\} \subset \rho(A)$, *then* $A$ *has a* $(\alpha, \beta)$ *exponential dichotomy.*

## Spectral mapping theorem $\sigma(e^{At}) = \overline{e^{\sigma(A)t}}$



## Proof of spectral mapping theorem

- High frequency estimates of spectrum and resolvent parallel to the imaginary axis:
  - For high frequencies spectrum and resolvent are approximated by the diagonal system.
  - For $\lambda$ on stripes in the resolvent set parallel to the imaginary axis we have for $|\mathfrak{Im}\,\lambda|$ sufficiently large

$$R(\lambda, A) = R(\lambda, A_0) + \frac{1}{\lambda} R_1(\lambda) + O\left(\frac{1}{\lambda^2}\right).$$

  - $A_0$ denotes the diagonal system, obtained by cancelling all nondiagonal entries in the linearized differential equation. Since equations decouple there is a closed formula for $R(\lambda, A_0)$.
  - Error term $R_1(\lambda)$ as well as higher order terms can be calculated recursively (terms quite complicated).

## Estimates for spectrum

### Theorem

*There exists an exponential polynomial* $h_0$ *and an entire (characteristic) function* $h$ *with the following properties:*

- $\sigma(A) = \{\lambda \in \mathbb{C} \mid h(\lambda) = 0\}$,
- $\sigma(A_0) = \{\lambda \in \mathbb{C} \mid h_0(\lambda) = 0\}$,
- *For all* $r > 0$ *there exist* $c, d > 0$ *such that for all* $\lambda \in \mathbb{C}$ *with* $|\mathfrak{Re}\,\lambda| < r$ *und* $|\mathfrak{Im}\,\lambda| > d$ *we have:*

$$\left| h(\lambda) - h_0(\lambda) - \frac{1}{\lambda} h_1(\lambda) \right| \leq c \frac{1}{|\lambda|^2},$$

- *There is a closed formula for* $h_1$ *(quite complicated).*

## Resolvent estimates

### Theorem

*Let $U \subset \rho(A)$ be such that*

$$\sup_{\lambda \in U} |\mathfrak{Re}\,\lambda| < \infty, \quad \inf_{\lambda \in U} |h_0(\lambda)| > 0.$$

*Then there exists $d > 0$ such that for $\lambda \in U$ with $|\mathfrak{Im}\,\lambda| \geq d$*

- $R(\lambda, A) = R(\lambda, A_0) + \frac{1}{\lambda} R_1(\lambda, A) + \frac{1}{\lambda^2} \mathcal{E}(\lambda, A)$,
- $R(\lambda, A_0)$, $R_1(\lambda, A)$ *and* $\mathcal{E}(\lambda, A)$ *are bounded on* $U$,
- *There are closed formulas for* $R_1(\lambda, A)$ *and* $R(\lambda, A_0)$.

- In particular the resolvent $R(\lambda, A)$ is bounded on $U$.

## Spectral mapping theorem in $L^2$

- By applying an important theorem due to Gearhart/Herbst/Prüss [Trans. AMS 1984] the resolvent estimates imply the following spectral mapping property for linearized hyperbolic systems in the Hilbert space $L^2$

$$\sigma(e^{At}) \setminus \{0\} = \overline{e^{\sigma(A)t}} \setminus \{0\}.$$

- Problem: theorem of Gearhart/Herbst/Prüß requires Hilbert-space.
- The semiflow is not strongly linearizable in $L^2$.
- We need a spectral mapping theorem or characterization of exponential dichotomy in terms of the spectrum of $A$ in the smaller nonreflexive Banach-space $X^{\mathbb{C}}$.

## Spectral mapping property in non Hilbert-space $X^{\mathbb{C}}$

- For Banach-space the situation is more difficult. Counterexamples show that Gearhart/Herbst/Prüss spectral mapping theorem fails, in general.
- Idea: Use $C_1$ Laplace-inversionformula. For $\rho > \omega(A)$

$$
\begin{aligned}
e^{At}x &= \frac{1}{2\pi \mathrm{i}} \,{}^{C_1}\!\!\int_{\rho-\mathrm{i}\infty}^{\rho+\mathrm{i}\infty} e^{zt} R(z,A)x\, dz \\
&:= \frac{1}{2\pi} \lim_{R\to\infty} e^{\rho t} \int_{-R}^{R} e^{\mathrm{i}\nu t} R(\rho+\mathrm{i}\nu, A)x \left(1 - \frac{|\nu|}{R}\right) d\nu.
\end{aligned}
$$

- Works also in non Hilbert-space $X^{\mathbb{C}}$ !

## Characterization of $(\alpha, \beta)$ dichotomy in $X^{\mathbb{C}}$

### Theorem (1994 Lunel, Kaashoek in J. Diff. Eq.)

*$A$ has a $(\alpha, \beta)$ exponential dichotomy if and only if*

$\imath)$ $\rho(A) \supset \{\lambda \in \mathbb{C} \mid \alpha < \mathfrak{Re}\,\lambda < \beta\}$,

$\imath\imath)$ *For all* $\delta > 0$ : $\sup\limits_{\alpha+\delta<\mathfrak{Re}\lambda<\beta-\delta} \|R(\lambda, A)\| < \infty$,

$\imath\imath\imath)$ *For all* $\rho \in \,]\alpha, \beta[$ *there exists a constant* $K_\rho > 0$ *such that for all* $x \in X^{\mathbb{C}}, x^* \in \left(X^{\mathbb{C}}\right)^*$

$\mathfrak{F}r(\cdot, \rho, x, x^*) \in L^\infty(\mathbb{R})$ *and* $\|\mathfrak{F}r(\cdot, \rho, x, x^*)\|_{L^\infty} \leq K_\rho \|x\| \|x^*\|$,

*where* $r(\cdot, \rho, x, x^*) : \mathbb{R} \to \mathbb{C}$ *is defined as*

$$r(\nu, \rho, x, x^*) := x^* R(\rho + i\nu, A)x.$$

## Characterization of $(\alpha, \beta)$ dichotomy in $X^{\mathbb{C}}$

- Necessity of conditions follows directly from the Hille-Yosida theorem and the $C_1$ Laplace inversion formula applied to $A_{|\mathrm{Im}\,\pi}$ and $A_{|\mathrm{Ker}\,\pi}$.
- Sufficiency: Proved in two papers by Kaashoek, Lunel, Latushkin [J. Diff. Eq. 1992, Oper. Theor. Adv. Appl. 2001].
- The resolvent estimates are in sufficiently closed form so that the Fourier transforms can be estimated.
- Warning: Convergence of improper Fourier integrals only in Cesaro mean $C_1$, no absolute convergence.
- Tools:
  - $\frac{1}{2\pi} \overset{C_1}{\int_{\mathbb{R}}} e^{i\omega t} \mathfrak{F}^{-1} f(\omega)\, d\omega = \frac{f(t+)+f(t-)}{2}$,
  - Wiener Algebra property of absolutely convergent Fourier series.

**Theorem**

*Principle of linearized stability and center manifold theorem hold true for hyperbolic systems.*

## Applications

The results are applicable to large classes of practical problems:

- ▷ Stability and bifurcation analysis in Laser dynamics
- ▷ Model Reduction: Mode approximations [Bandelow, Wenzel, Wünsche 1993]
- ▷ Turing-Models with correlated random walk [Kac, Goldstein, Hadeler, Hillen, Horsthemke, ...]
- ▷ Boltzmann-systems
- ▷ Tubular reactor processes
- ▷ Systems of vibrating strings
- ▷ Differential equations with delay
- ▷ ...

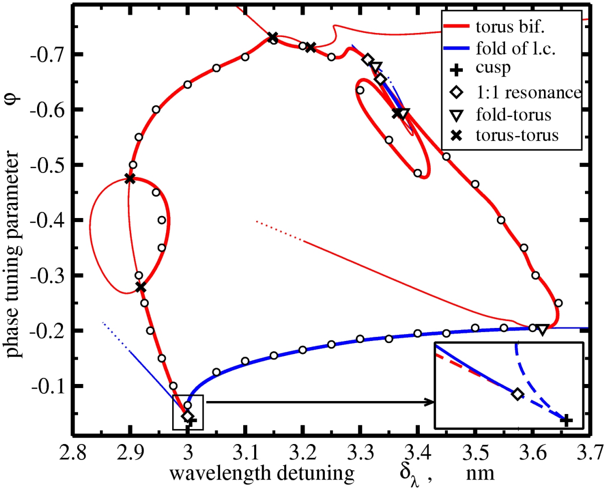
## The traveling wave model

$$\begin{aligned}
\frac{1}{v_g}\partial_t E^{\pm} &= (\mp\partial_z - i\beta(n))\, E^{\pm} - i\kappa E^{\mp} - \frac{\overline{g}}{2}\left(E^{\pm} - P^{\pm}\right) \\
\partial_t P^{\pm} &= \overline{\gamma}\left(E^{\pm} - P^{\pm}\right) + i\overline{\omega} P^{\pm} \\
\partial_t n &= I - R(n) - v_g \mathfrak{Re}\langle E, g(n)E - \overline{g}(E-P)\rangle_{\mathbb{C}^2}
\end{aligned}$$

$$E^{+}(t,0) = r_0 E^{-}(t,0) + \alpha(t), \quad E^{-}(t,l) = r_l E^{+}(t,l).$$

- $t \in \mathbb{R}$ time, $z \in [0,l]$ longitudinal coordinate
- $E^{\pm} = E^{\pm}(t,z) \in \mathbb{C}$ complex envelope of optical field, $P^{\pm} = P^{\pm}(t,z) \in \mathbb{C}$ polarization, $n = n(t,z) \in \mathbb{R}$ carrier density
- spontaneous recombination $R(n) = An + Bn^2 + Cn^3$
- propagation coefficient $\beta(n) = \delta_0 - i\frac{\alpha}{2} + \frac{i}{2}g(n) + \delta_N(n)$
- field gain $g(n) = G' \log \frac{n}{n_{tr}}$, effective index dependence $\delta_N(n) = -\sqrt{n'n}$, current injection $I = I(t,z)$, optical injection $\alpha(t)$ at left facet of laser, reflection coeffecients $r_0$ and $r_l$
- all coefficients depend on lateral coordinate $z$

## A two parameter numerical bifurcation analysis of the raveling wave model (LDSL, M. Radziunas)



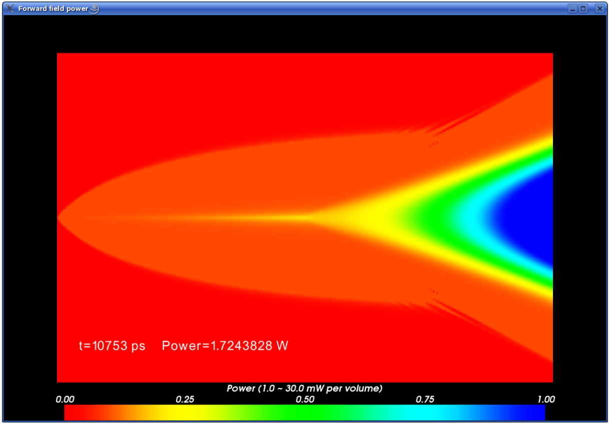
## Current Work: The 2d diffraction extended traveling wave model

$$\begin{aligned}
\frac{1}{v_g}\partial_t E^\pm &= -i\frac{1}{2K}\partial_{xx}E^\pm + (\mp\partial_z - i\beta(n))\,E^\pm - i\kappa E^\mp - \frac{\overline{g}}{2}\left(E^\pm - P^\pm\right) \\
\partial_t P^\pm &= \overline{\gamma}\left(E^\pm - P^\pm\right) + i\overline{\omega}P^\pm \\
\partial_t n &= d_n\partial_{xx}n + I - R(n) - v_g\mathfrak{Re}\langle E, g(n)E - \overline{g}(E-P)\rangle_{\mathbb{C}^2}
\end{aligned}$$

$$E^+(t,0,x) = r_0(x)E^-(t,0,x) + \alpha(t,x), \quad E^-(t,l,x) = r_l(x)E^+(t,l,x).$$

- All coefficients now depend on longitudinal ($z$) and lateral ($x$) coordinate.
- In optical equation for $E$ a diffraction operator has been addd (red).
- In carrier equation for $n$ a lateral diffusion operator has been added (blue).

## Stable pulsating high power diode




Laser geometry and parameters provided by FBH.